



IPIN 2018



# TUTORIAL

If you tell me what you  
look at, I could tell you  
where you are

by

**Dr. Myriam Servières**  
(École Centrale de Nantes)



**IPIN 2018**

NINTH INTERNATIONAL CONFERENCE ON  
INDOOR POSITIONING  
AND INDOOR NAVIGATION

SEPTEMBER 24-27, 2018, NANTES, FRANCE



TEAM  
GEOLOC



IFSTAR

If you tell me what you look at, I could tell you  
where you are

Some state of the art pose estimation methods in computer vision

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École Centrale Nantes, AAU-CRENAU, IRSTV

École Centrale Nantes, AAU-CRENAU, IRSTV



Nantes - 24 september 2018

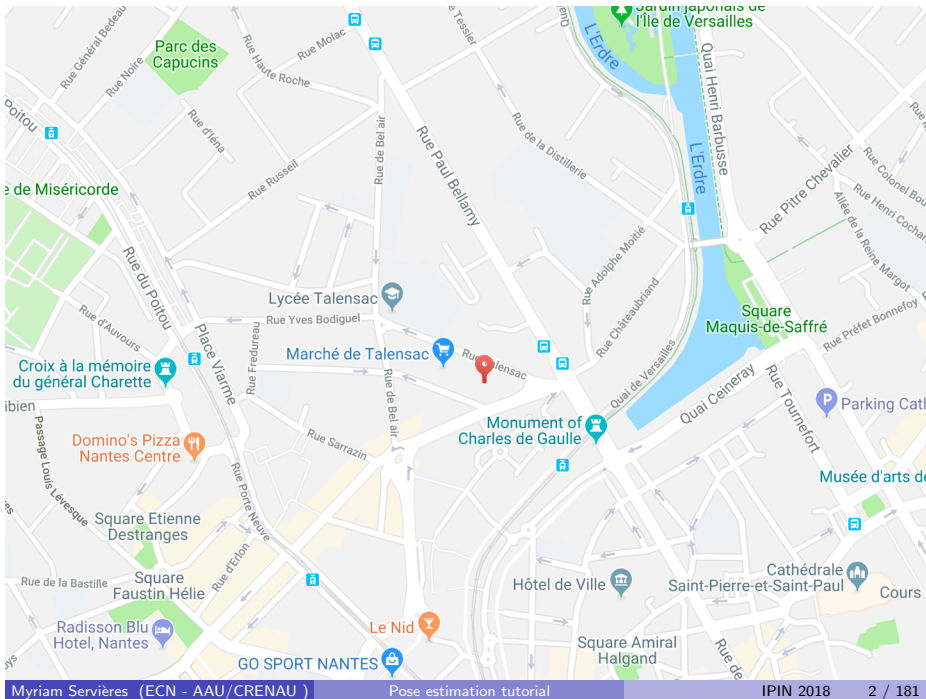


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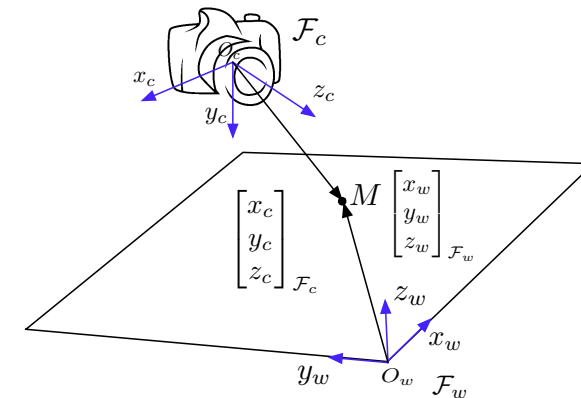


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## Camera Pose



### Camera pose

Camera position and orientation relative to a fixed coordinate system

## Definitions

### Camera pose

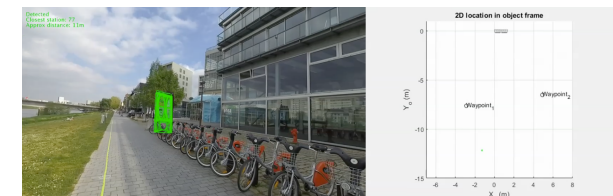
Camera position and orientation relative to a fixed coordinate system

### Computer Vision (CV)

Interdisciplinary field that study and develop techniques to enable a computer system or artificial intelligence system to analyze and understand visual data obtained using cameras. Computer vision tasks include methods for acquiring, processing, analyzing and understanding digital images.

## Why do we need the pose for?

- Retrieving information from image

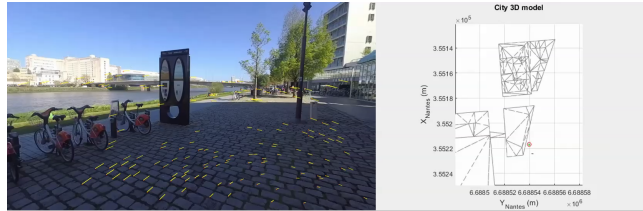


work from [ASR17]

- Knowing a displacement only from a video
- Augmented Reality
- Reconstructing a 3D model
- ...

## Why do we need the pose for?

- Retrieving information from image
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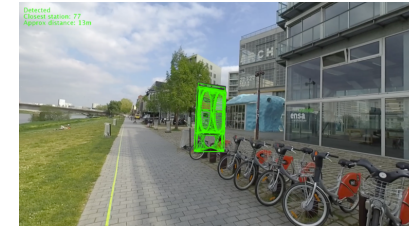


work from [ASR18]

- Augmented Reality
- Reconstructing a 3D model
- ...

## Why do we need the pose for?

- Retrieving information from image
- Knowing a displacement only from a video
- Augmented Reality

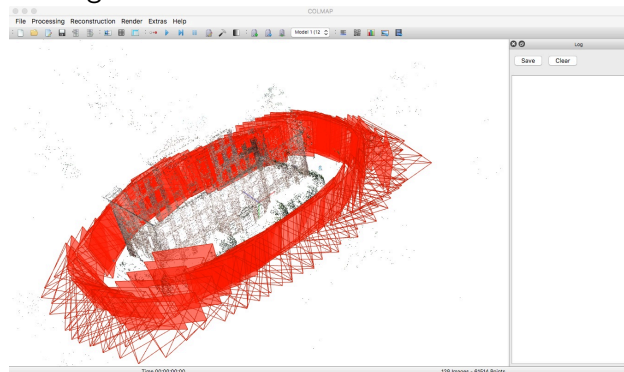


work from [ASR17]

- Reconstructing a 3D model
- ...

## Why do we need the pose for?

- Retrieving information from image
- Knowing a displacement only from a video
- Augmented Reality
- Reconstructing a 3D model



3D model reconstructed with COLMAP with an example dataset [SF16, SZPF16]

<https://demuc.de/colmap/#tutorial>

## Outline

- 1 Camera model and calibration
- 2 Some pose estimation algorithms with a known 3D model
- 3 Transformation between images
- 4 Motion estimation
- 5 Conclusion



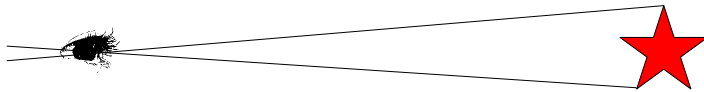
## Outline

- 1 Camera model and calibration
  - Pinhole Camera Model
  - Calibration parameters
  - Calibration methods
  - References, tools and demo
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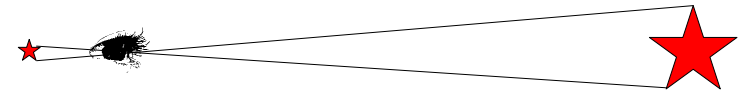
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## Pinhole Camera Model



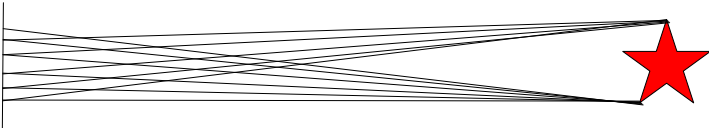
## Pinhole Camera Model



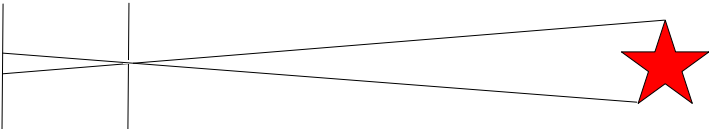
# Pinhole Camera Model



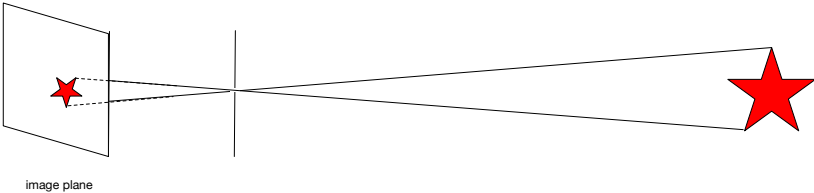
# Pinhole Camera Model



# Pinhole Camera Model

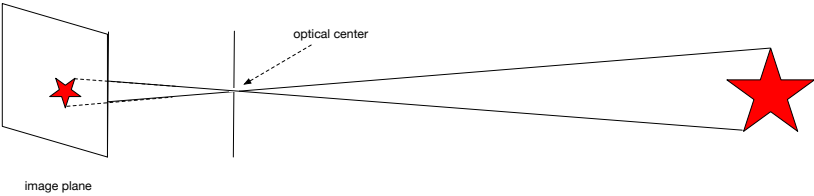


# Pinhole Camera Model

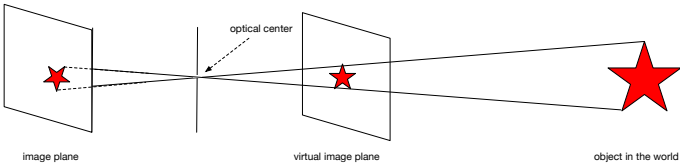




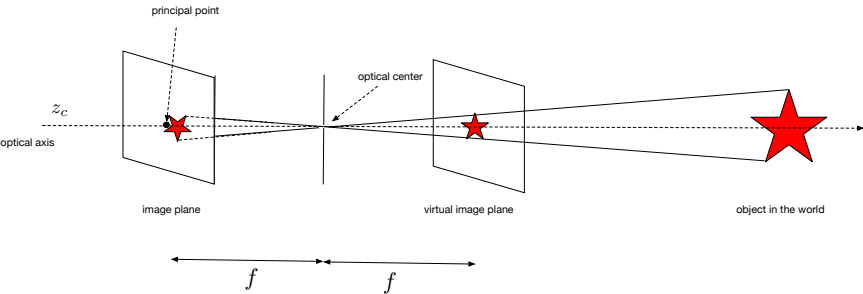
# Pinhole Camera Model



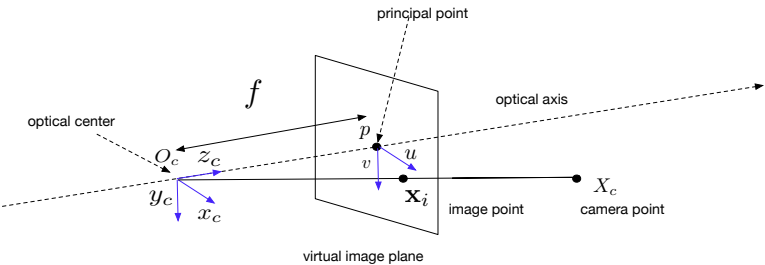
# Pinhole Camera Model



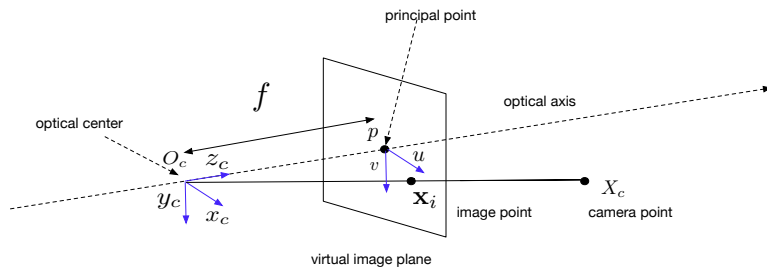
# Pinhole Camera Model



# Pinhole Camera Model



# Pinhole Camera Model



- Model of the camera's geometry
- Distortion model of the lens

# Outline

## 1 Camera model and calibration

- Pinhole Camera Model
- Calibration parameters
  - Link between imaged point and point on image
  - Transformations using homogeneous coordinates
  - Extrinsic parameters
  - Camera matrix
  - Lens distortions
- Calibration methods
- References, tools and demo

- 2 Some pose estimation algorithms with a known 3D model
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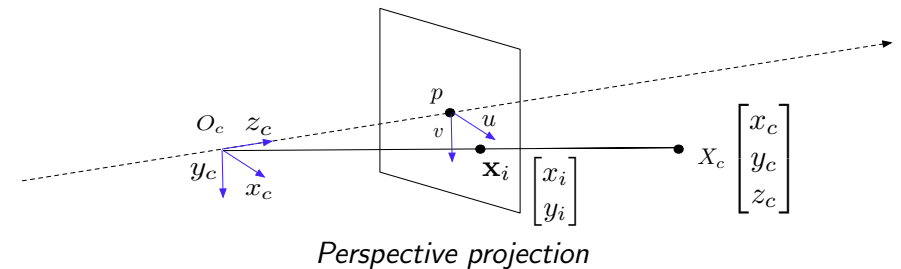
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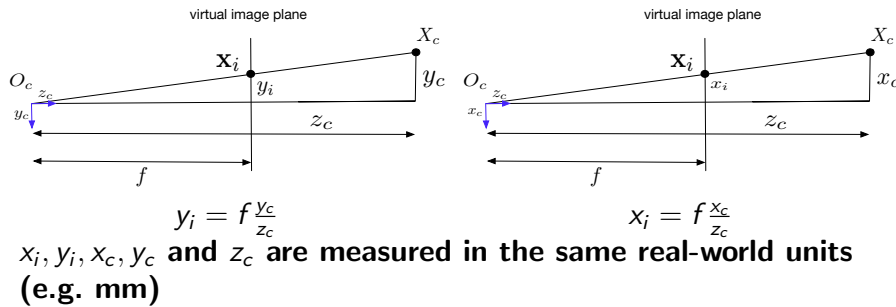
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# Link between imaged point and point on image

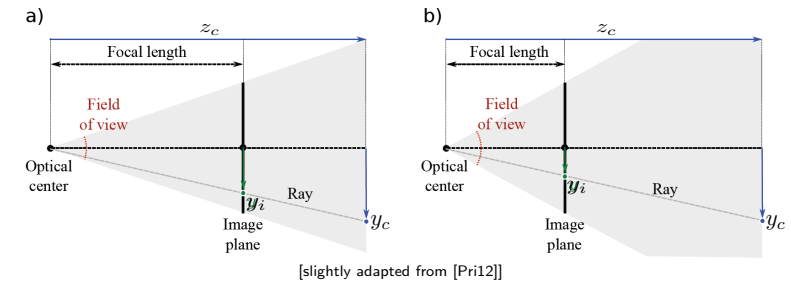




## Link between imaged point and point on image



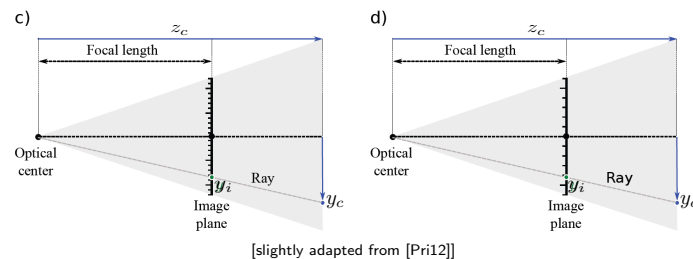
## Focal length and photoreceptor spacing



## Camera field of view

Total angular range that is imaged (usually different in the x- and y-directions)

## Focal length and photoreceptor spacing



Unit in pixel on the image

## Rectangular pixels

- two different focal lengths :  $f_x$  and  $f_y$
- $f_x = f \cdot s_x$  and  $f_y = f \cdot s_y$  with
  - ▶  $f$  the physical focal length in the chosen real-world unit (e.g. mm)
  - ▶  $s_x$  (resp.  $s_y$ ) the size of the individual imager elements along  $x$  (resp.  $y$ ) in px/chosen real-world unit (e.g. px/mm)
- Only  $f_x$  and  $f_y$  can be derived by the calibration process

$$u = f_x \frac{x_c}{z_c}$$

$$V = f_y \frac{y_c}{z_c}$$

## Representation in homogeneous coordinates

- $X_c = [x_c, y_c, z_c]^T$  correspond to  $\mathbf{x}_{ipix} = [u, v]^T = [f_x \frac{x_c}{z_c}, f_y \frac{y_c}{z_c}]^T$
- non-linear operation ( $/z_c$ )  $\rightarrow$  perspective projection
- Use homogeneous coordinates to get a linear form (add one dimension).

$$\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \rightarrow \begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix}$$

- conversion from homogeneous coordinates

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \rightarrow \begin{bmatrix} u/w \\ v/w \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \rightarrow \begin{bmatrix} x/w \\ y/w \\ z/w \end{bmatrix}$$

## Representation in homogeneous coordinates

- Using homogeneous coordinates

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f_x \frac{x_c}{z_c} \\ f_y \frac{y_c}{z_c} \end{bmatrix}$$

can be expressed in a matrix form

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \frac{1}{z_c} \begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \text{ or } \lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

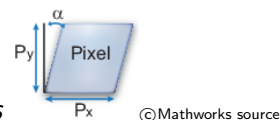
with  $\lambda$  a scale factor

## Offset and skew parameters

- Principal point  $p$  is not at  $[0, 0]^T$  in  $\mathcal{F}_i$
- Point  $[0, 0]^T$  is at upper-left corner and  $p = [u_0, v_0]^T$

$$u = f_x \frac{x_c}{z_c} + u_0$$

$$v = f_y \frac{y_c}{z_c} + v_0$$



- Pixels may have a skew factor  $s$

$$u = f_x \frac{x_c + s \cdot y_c}{z_c} + u_0$$

$$v = f_y \frac{y_c}{z_c} + v_0$$

## Offset and skew parameters

- Full pinhole camera model in a matrix form

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & s & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

(linear!)



## Intrinsic parameters matrix

$$\mathbf{K} = \begin{bmatrix} f_x & s & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix}$$

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### 2 Some pose estimation algorithms with a known 3D model

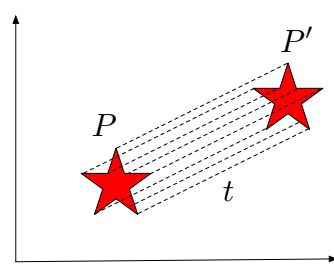
### 3 Transformation between images

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## 2D translation



$$P = [x, y]^T \rightarrow [x, y, 1]^T$$

$$t = [t_x, t_y]^T \rightarrow [t_x, t_y, 1]^T$$

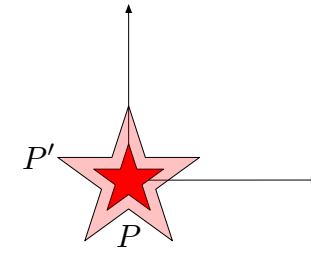
$$P' = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$$

$$P = \mathbf{T}.P$$

2 D.o.F

## 2D scale change



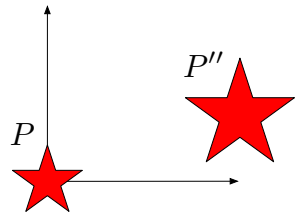
$$P = [x, y]^T \rightarrow [x, y, 1]^T$$

$$P' = [s_x \cdot x, s_y \cdot y]^T \rightarrow [s_x \cdot x, s_y \cdot y, 1]^T$$

$$P' = \begin{bmatrix} s_x \cdot x \\ s_y \cdot y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} \mathbf{s} & 0 \\ 0 & 1 \end{bmatrix} P = \mathbf{S}.P$$

## 2D scale change and translation



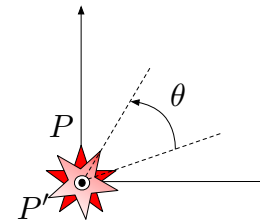
$$P'' = \mathbf{T}.P' = \mathbf{T}.\mathbf{S}.P$$

note :  $\mathbf{T}.\mathbf{S}.P \neq \mathbf{S}.\mathbf{T}.P$

$$\mathbf{T}.\mathbf{S}.P = \begin{bmatrix} s_x \cdot x + t_x \\ s_y \cdot y + t_y \\ 1 \end{bmatrix} \text{ and}$$

$$\mathbf{S}.\mathbf{T}.P = \begin{bmatrix} s_x \cdot x + s_x \cdot t_x \\ s_y \cdot y + s_y \cdot t_y \\ 1 \end{bmatrix}$$

## 2D rotation



$$P' = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

1 D.o.F.

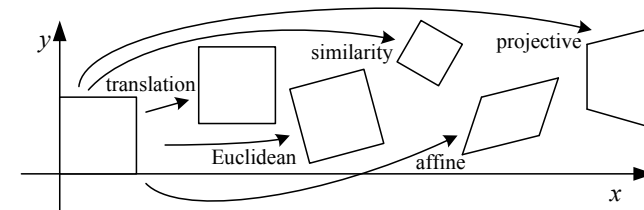
## 2D similarity

- Similarity = scale (with  $s_x = s_y$ ) + rotation + translation

$$P' = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

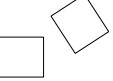

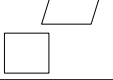
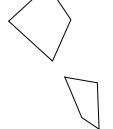
$$P' = \begin{bmatrix} \mathbf{RS} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

## 2D transformation summary



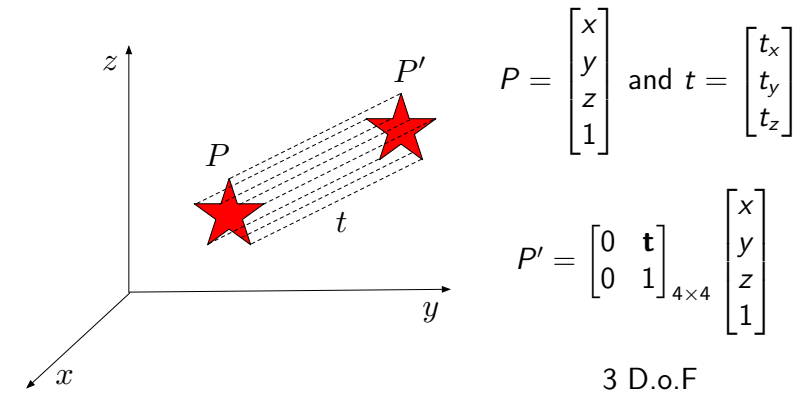
Extracted from [Sze10].

## 2D transformation summary

type	D.o.F.	matrix	transformed square	invariants
Euclidean	3	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		lengths, angles, parallelism, straight lines
Similarity	4	$\begin{bmatrix} s.r_{11} & s.r_{12} & t_x \\ s.r_{21} & s.r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		angles, parallelism, straight lines
Affine	6	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		parallelism, straight lines
Projective	8	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		straight lines

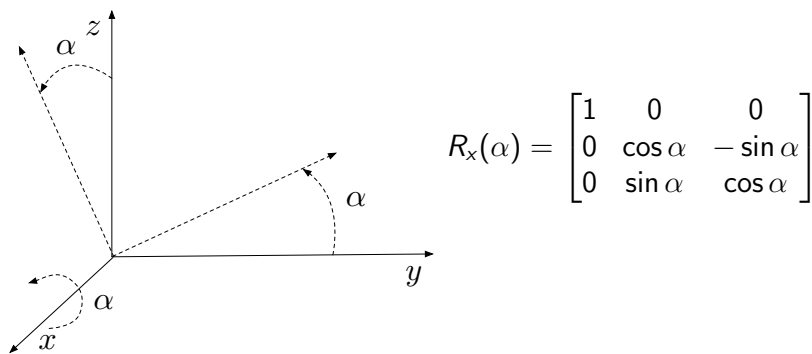
Adapted from [Sze10, HZ04]

## 3D translation



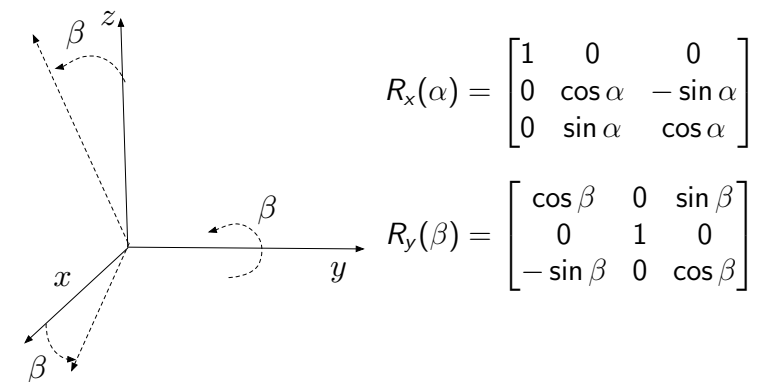
## 3D rotation

Counter-clockwise rotation around coordinate axes



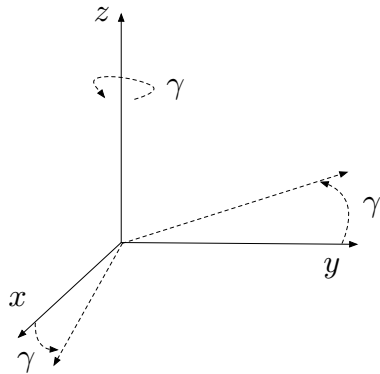
## 3D rotation

Counter-clockwise rotation around coordinate axes



## 3D rotation

Counter-clockwise rotation around coordinate axes



$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## 3D rotation

$$P' = \begin{bmatrix} \mathbf{R} & 0 \\ 0 & 1 \end{bmatrix}_{4 \times 4} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \text{ with } \mathbf{R} = R_z(\gamma)R_y(\beta)R_x(\alpha) \text{ or}$$

$$\mathbf{R} = \begin{bmatrix} \cos \beta \cos \gamma & \sin \alpha \sin \beta \cos \gamma - \cos \alpha \sin \gamma & \cos \alpha \sin \beta \cos \gamma + \sin \alpha \sin \gamma \\ \cos \beta \sin \gamma & \sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \gamma & \cos \alpha \sin \beta \sin \gamma - \sin \alpha \cos \gamma \\ -\sin \beta & \sin \alpha \cos \beta & \cos \alpha \cos \beta \end{bmatrix}$$

$$\text{Simplified notation : } \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\text{Reminder : } R^T R = R R^T = I$$

## 3D transformation summary

type	D.o.F.	matrix	transformed cube	invariants
Euclidean	6	$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$		lengths, angles, parallelism, straight lines
Similarity	7	$\begin{bmatrix} s.r_{11} & s.r_{12} & s.r_{13} & t_x \\ s.r_{21} & s.r_{22} & s.r_{23} & t_y \\ s.r_{31} & s.r_{32} & s.r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$		angles, parallelism, straight lines
Affine	12	$\begin{bmatrix} a_{11} & a_{12} & a_{13} & t_x \\ a_{21} & a_{22} & a_{23} & t_y \\ a_{31} & a_{32} & a_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$		parallelism, straight lines
Projective	15	$\begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{bmatrix}$		straight lines

Adapted from [Sze10, HZ04]

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### 1 Camera model and calibration

- Pinhole Camera Model
- Calibration parameters
  - Link between imaged point and point on image
  - Transformations using homogeneous coordinates
- Extrinsic parameters
  - Camera matrix
  - Lens distortions
- Calibration methods
- References, tools and demo

### 2 Some pose estimation algorithms with a known 3D model

### 3 Transformation between images

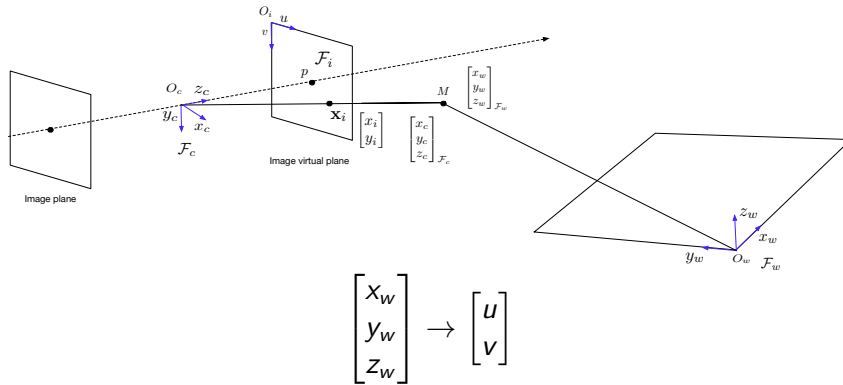
### 4 Motion estimation

### 5 Conclusion

### 6 Links and bibliography

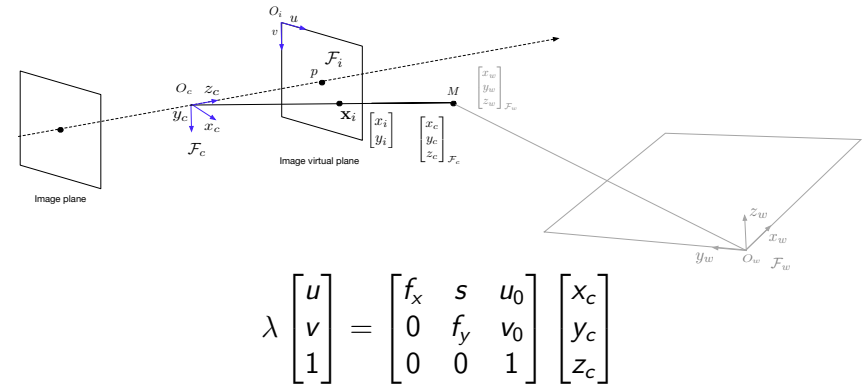
## Extrinsic parameters

What we want:



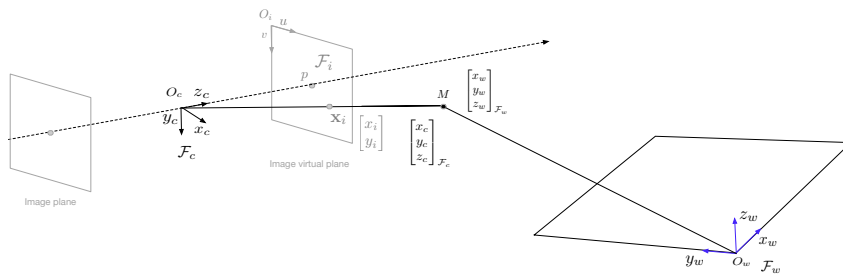
## Extrinsic parameters

What we have:



## Extrinsic parameters

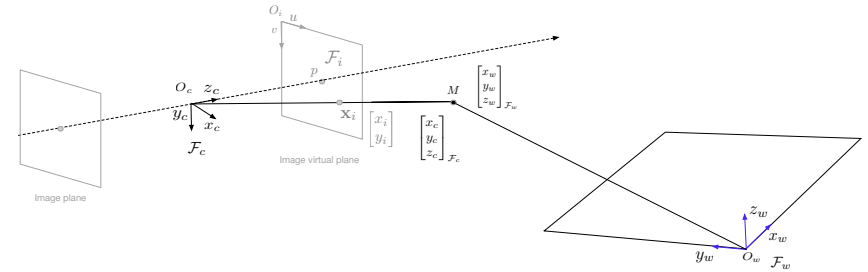
What is missing:



Position and orientation of the camera frame  $\mathcal{F}_c$   
in the world frame  $\mathcal{F}_w$ .

## Extrinsic parameters

What is missing:



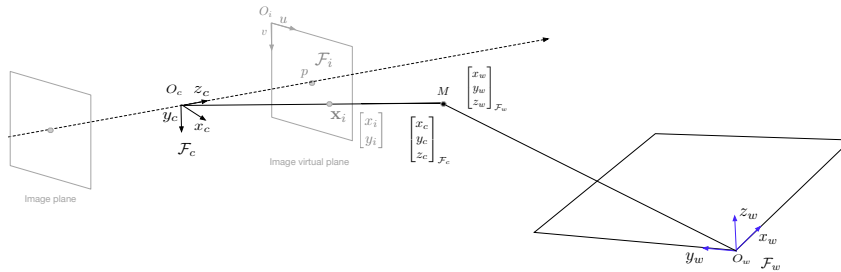
Position and orientation of the camera frame  $\mathcal{F}_c$   
in the world frame  $\mathcal{F}_w$ .

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$



## Extrinsic parameters

What is missing:



Position and orientation of the camera frame  $\mathcal{F}_c$  in the world frame  $\mathcal{F}_w$ .

$$\begin{bmatrix} x_c \\ y_c \\ z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix} \text{ with } \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \text{ extrinsic parameters matrix.}$$

## Outline

### 1 Camera model and calibration

- Pinhole Camera Model
- Calibration parameters
  - Link between imaged point and point on image
  - Transformations using homogeneous coordinates
  - Extrinsic parameters
  - Camera matrix
  - Lens distortions
- Calibration methods
- References, tools and demo

### 2 Some pose estimation algorithms with a known 3D model

### 3 Transformation between images

### 4 Motion estimation

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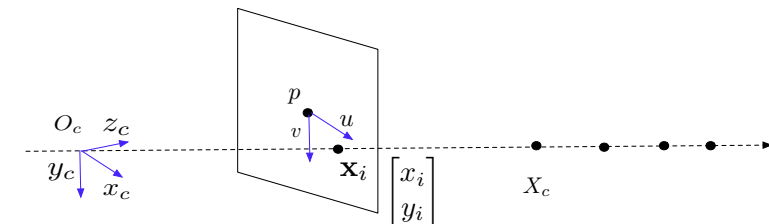
### 6 Links and bibliography

## Camera matrix

Finally we get :

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

## Camera matrix



All the 3D points on the same line passing through  $\mathbf{x}_i$  have the same image point.

## Camera matrix

$$\lambda \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

- $\lambda$  is a scale factor.
- $\mathbf{P} = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$  is the camera matrix.
- $\mathbf{K}$  5 D.o.F. and  $\mathbf{R}$  3 D.o.F. ,  $\mathbf{t}$  3 D.o.F.  
 $\Rightarrow$  11 D.o.F. in total (+1 for the scale)

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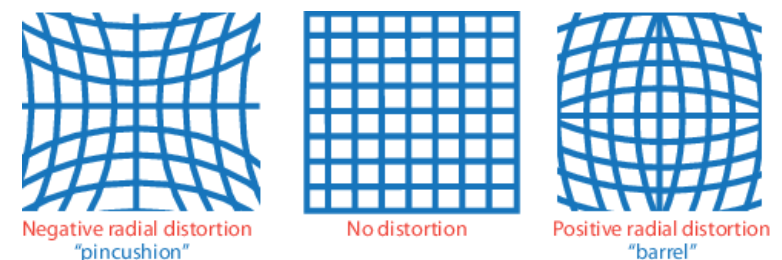
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## Lens distortions

- Real-world cameras are not pinhole: use of lenses
- Lenses introduce distortions:
  - ▶ Radial distortions
  - ▶ Tangential distortions



## Lens distortions - Radial distortion



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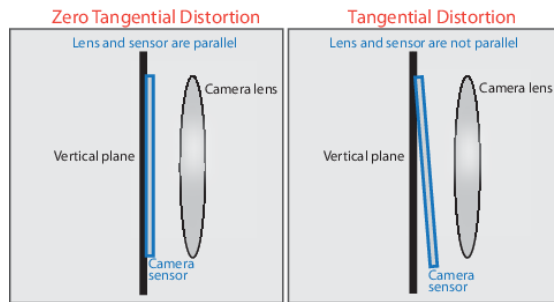
$$x_{final} = x(1 + k_1 * r^2 + k_2 * r^4 + k_3 * r^6)$$

$$y_{final} = y(1 + k_1 * r^2 + k_2 * r^4 + k_3 * r^6)$$

with  $k_1, k_2, k_3$  distortion coefficients and  $r$  distance to optical center

Note: This distortion is implemented after perspective projection but before the effect of the intrinsic parameters so the warping is relative to the optical axis and not the origin of the pixel coordinate system.

## Lens distortions - Tangential distortions



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$$\begin{aligned}x_{final} &= x + [2p_1y + p_2(r^2 + 2x^2)] \\y_{final} &= y + [p_1(r^2 + 2y^2) + 2p_2x]\end{aligned}$$

with  $p_1, p_2$  tangential distortion parameters.

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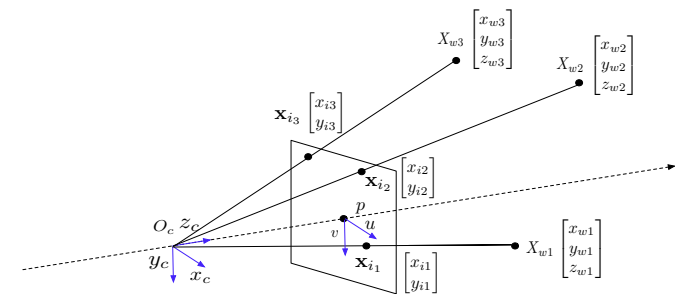
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## Linear approach principle

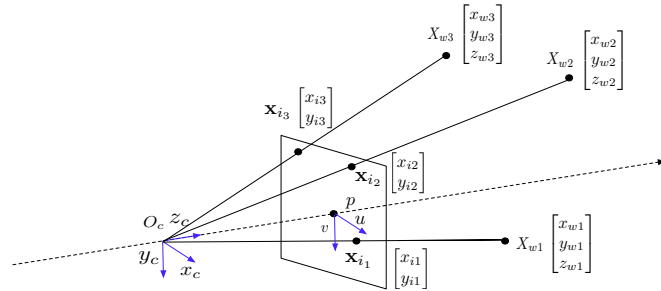


Knowing 3D points  $X_{w_i}$  and their corresponding points in the image  $X_{i_i}$

$P$ - $n$ - $P$ : Perspective- $n$ -point

- Estimate  $\mathbf{P}$
- Extract  $\mathbf{K}$ ,  $\mathbf{R}$  and  $\mathbf{t}$  from  $\mathbf{P}$

## Linear approach principle



How many corresponding points are needed?

- $\mathbf{P} = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}$  as 12 entries and 11 D.o.F. (ignoring the scale)
- each correspondence point leads to 2 eq.  
 $\Rightarrow$  5 correspondences + 1 eq. are needed: 6 correspondence points

## Direct Linear Transform: $\mathbf{x}_i \leftrightarrow \mathbf{X}_w$

- Considering  $\mathbf{P}_{3 \times 4}$  and  $\mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$  a point in the image

corresponding to a 3D point  $\mathbf{X}_w = \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$

- $\mathbf{x}_i = \mathbf{P}\mathbf{X}_w$  for all corresponding points
- We have the linear relationship:

$$\mathbf{x}_i \times \mathbf{P}\mathbf{X}_w = 0 \text{ (with } \times \text{ cross product)}$$

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## Direct Linear Transform: $\mathbf{x}_i \times \mathbf{P}\mathbf{X}_w = 0$

- writing  $\mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$  with  $P_i$  a 4-vector of the  $i$ -th row of  $\mathbf{P}$
- then  $\mathbf{P}\mathbf{X}_w = \begin{bmatrix} P_1^T \cdot \mathbf{X}_w \\ P_2^T \cdot \mathbf{X}_w \\ P_3^T \cdot \mathbf{X}_w \end{bmatrix}$
- and  $\mathbf{x}_i \times \mathbf{P}\mathbf{X}_w = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \times \begin{bmatrix} P_1^T \cdot \mathbf{X}_w \\ P_2^T \cdot \mathbf{X}_w \\ P_3^T \cdot \mathbf{X}_w \end{bmatrix}$

## Direct Linear Transform: $\mathbf{x}_i \times \mathbf{P} \mathbf{X}_w = 0$

- $\mathbf{x}_i \times \mathbf{P} \mathbf{X}_w = 0$
- Developing it:
 
$$\begin{bmatrix} x_i P_2^T X_w - y_i P_1^T X_w \\ -x_i P_3^T X_w + z_i P_1^T X_w \\ y_i P_3^T X_w - z_i P_2^T X_w \end{bmatrix} = 0$$
- As  $P_i^T X_w = X_w^T P_i$ :
 
$$\begin{bmatrix} x_i X_w^T P_2 - y_i X_w^T P_1 \\ -x_i X_w^T P_3 + z_i X_w^T P_1 \\ y_i X_w^T P_3 - z_i X_w^T P_2 \end{bmatrix} = 0$$
- and writing in a matrix form:
 
$$\begin{bmatrix} 0 & -z_i X_w^T & y_i X_w^T \\ z_i X_w^T & 0 & -x_i X_w^T \\ -y_i X_w^T & x_i X_w^T & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$

## Direct Linear Transform: determination of $\mathbf{P}$

- the 3 rows are linearly dependent
 
$$\begin{bmatrix} 0 & -z_i X_w^T & y_i X_w^T \\ z_i X_w^T & 0 & -x_i X_w^T \\ -y_i X_w^T & x_i X_w^T & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$
- So we choose only the 2 first ones
 
$$\begin{bmatrix} 0 & -z_i X_w^T & y_i X_w^T \\ z_i X_w^T & 0 & -x_i X_w^T \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = 0$$
- this will be written  $A_i \mathbf{p} = 0$  with  $\mathbf{p}$  a 12-vector made up of the entries of  $\mathbf{P}$ .
- note: The eq. hold for any homogeneous representation of  $\mathbf{x}_i$ .  
Choosing  $z_i = 1$ , then  $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$  are coordinate on the image

## Direct Linear Transform: determination of $\mathbf{P}$

- Staking all eq. for each correspondence point leads to:  
 $A \mathbf{p} = 0$  with  $A$  matrix of eq. coefficients built from the matrix rows  $A_i$  and  $\mathbf{p}$  and 12-vector made up of the entries of the matrix  $\mathbf{P}$
- Obtain the SVD of  $A$ . The unit singular vector corresponding to the smallest singular value is the solution of  $\mathbf{p}$ .  
Specifically, if  $A = UDV^T$  with  $D$  diagonal with positive diagonal entries, arranged in descending order down the diagonal, then  $\mathbf{p}$  is the last column of  $V$ .
- $\mathbf{P}$  is then given by  $\mathbf{p}$

[DLT algorithm from [HZ04]].

## Direct Linear Transform: determination of $\mathbf{P}$

- Minimal solution  
Given 6 correspondences, the solution is exact.  
The solution is obtained solving  $A \mathbf{p} = 0$  where  $A$  is  $11 \times 12$ .  
In general  $A$  will have rank 11, and the solution vector  $\mathbf{p}$  is the 1-dimensional right null-space of  $A$ .
- Over-determined solution  
If data are not exact (noise) and  $n \geq 6 \rightarrow$  not an exact solution to  $A \mathbf{b} = 0$ .  
Then the estimation of  $\mathbf{P}$  may be obtained by minimizing an algebraic error with the normalization constraint  $\|\mathbf{p}\| = 1$

[From [HZ04]].



## DLT: points normalization

- DLT is not invariant to similarity transformations (details in [HZ04])  
⇒ apply a normalizing transformation to the data before applying the DLT algorithm
- Normalization for 2D points: Isotropic scaling
  - ▶ points are translated so that their centroid is at the origin.
  - ▶ points are then scaled so that their root-mean-square distance from the origin is equal to  $\sqrt{2}$
- Normalization for 3D points - case with small variations in points depth
  - ▶ centroid of the points is translated to the origin
  - ▶ coordinates of the points are scaled so that their root-mean-square distance from the origin is  $\sqrt{3}$

## Direct Linear Transform complete algorithm [HZ04]

Objective:

Given  $n \geq 6$  world to image point correspondences  $X_w \leftrightarrow \mathbf{x}_i$ , determine the camera projection matrix  $\mathbf{P}$  such as  $\mathbf{x}_i = \mathbf{P}X_w$

Algorithm

- 1 **Normalization of  $\mathbf{x}_i$ :** Compute a similarity transformation  $T$ , consisting of a translation and scaling, that takes points  $\mathbf{x}_i$  to a new set of points  $\tilde{\mathbf{x}}_i$  such that the centroid of the points  $\tilde{\mathbf{x}}_i$  is the coordinate origin  $[0, 0]^T$ , and their average distance from the origin is  $\sqrt{2}$ .

## Direct Linear Transform complete algorithm [HZ04]

Objective:

Given  $n \geq 6$  world to image point correspondences  $X_w \leftrightarrow \mathbf{x}_i$ , determine the camera projection matrix  $\mathbf{P}$  such as  $\mathbf{x}_i = \mathbf{P}X_w$

Algorithm

- 1 **Normalization of  $\mathbf{x}_i$ :** Compute a similarity transformation  $T$
- 2 **Normalization of  $X_w$ :** Compute a similarity transformation  $U$ , consisting of a translation and scaling, that takes points  $X_w$  to a new set of points  $\tilde{X}_w$  such that the centroid of the points  $\tilde{X}_w$  is the coordinate origin  $[0, 0, 0]^T$ , and their average distance from the origin is  $\sqrt{3}$ .

## Direct Linear Transform complete algorithm [HZ04]

Objective:

Given  $n \geq 6$  world to image point correspondences  $X_w \leftrightarrow \mathbf{x}_i$ , determine the camera projection matrix  $\mathbf{P}$  such as  $\mathbf{x}_i = \mathbf{P}X_w$

Algorithm

- 1 **Normalization of  $\mathbf{x}_i$ :** Compute a similarity transformation  $T$
- 2 **Normalization of  $X_w$ :** Compute a similarity transformation  $U$
- 3 **DLT:**
  - 1 For each correspondence  $\tilde{X}_w \leftrightarrow \tilde{\mathbf{x}}_i$  compute the matrix  $A_i$ . Only the first two rows need be used in general.
  - 2 Form the  $2n \times 12$  matrix  $A$  by stacking the equations generated by each correspondence  $\tilde{X}_w \leftrightarrow \tilde{\mathbf{x}}_i$ .
  - 3 Write  $\mathbf{p}$  for the vector containing the entries of the matrix  $\tilde{\mathbf{P}}$ . A solution of  $A\mathbf{p} = 0$ , subject to  $\|\mathbf{p}\| = 1$ , is obtained from the unit singular vector of  $A$  corresponding to the smallest singular value.

## Direct Linear Transform complete algorithm [HZ04]

Objective:

Given  $n \geq 6$  world to image point correspondences  $X_w \leftrightarrow \mathbf{x}_i$ , determine the camera projection matrix  $\mathbf{P}$  such as  $\mathbf{x}_i = \mathbf{P}X_w$

Algorithm

- 1 **Normalization of  $\mathbf{x}_i$ :** Compute a similarity transformation  $T$
- 2 **Normalization of  $X_w$ :** Compute a similarity transformation  $U$
- 3 **DLT:** Compute  $\mathbf{p}$  from  $A$  then get  $\tilde{\mathbf{P}}$
- 4 **Denormalization.** The camera matrix for the original (unnormalized) coordinates is obtained from  $\tilde{\mathbf{P}}$  as :

$$\mathbf{P} = T^{-1}\tilde{\mathbf{P}}U$$

Note: an implementation for leaning purposes can be found at

► <http://people.rennes.inria.fr/Eric.Marchand/pose-estimation/tutorial-pose-dlt-opencv.html>

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## Extraction of the extrinsic and intrinsic parameters

Having  $\mathbf{P}$ :

$$\mathbf{P} = \mathbf{K} \begin{bmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{bmatrix}$$

We can write:

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} f_x & s & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

Then:

$$\mathbf{P} = \begin{bmatrix} f_x r_{11} + s r_{21} + u_0 r_{31} & f_x r_{12} + s r_{22} + u_0 r_{32} & f_x r_{13} + s r_{23} + u_0 r_{33} & f_x t_1 + s t_2 + u_0 t_3 \\ f_y r_{21} + v_0 r_{31} & f_y r_{22} + v_0 r_{32} & f_y r_{23} + v_0 r_{33} & f_y t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

[From [Mar13]]

## Extraction of the extrinsic and intrinsic parameters

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} f_x r_{11} + s r_{21} + u_0 r_{31} & f_x r_{12} + s r_{22} + u_0 r_{32} & f_x r_{13} + s r_{23} + u_0 r_{33} & f_x t_1 + s t_2 + u_0 t_3 \\ f_y r_{21} + v_0 r_{31} & f_y r_{22} + v_0 r_{32} & f_y r_{23} + v_0 r_{33} & f_y t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

leads to:

- $r_{3i} = \frac{p_{3i}}{\|p_{3i}\|}$  for  $i = 1, 2, 3$
- $t_3 = p_{34}$

## Case $s = 0$ : get intrinsic parameters

$$\begin{bmatrix} f_x r_{11} + u_0 r_{31} & f_x r_{12} + u_0 r_{32} & f_x r_{13} + u_0 r_{33} & f_x t_1 + u_0 t_3 \\ f_y r_{21} + v_0 r_{31} & f_y r_{22} + v_0 r_{32} & f_y r_{23} + v_0 r_{33} & f_y t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix}$$

$$f_x = \left\| \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} - u_0 \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} \right\|$$

$$v_0 = \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix}$$

$$f_y = \left\| \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} - v_0 \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} \right\|$$

[From [Mar13]]

## Case $s = 0$ : get extrinsic parameters

$$\begin{bmatrix} f_x r_{11} + u_0 r_{31} & f_x r_{12} + u_0 r_{32} & f_x r_{13} + u_0 r_{33} & f_x t_1 + u_0 t_3 \\ f_y r_{21} + v_0 r_{31} & f_y r_{22} + v_0 r_{32} & f_y r_{23} + v_0 r_{33} & f_y t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

$$\begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} - u_0 \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} \cdot \frac{1}{f_x}$$

$$\begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} = \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} - v_0 \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} \cdot \frac{1}{f_y}$$

$$t_1 = \frac{p_{14} - u_0 \cdot t_3}{f_x}$$

$$t_2 = \frac{p_{24} - v_0 \cdot t_3}{f_y}$$

Notes: we must ensure  $\|r_i\| = 1$  with  $r_i$   $i$ -th column of  $R$  and we are not sure that  $r_3 = r_1 \times r_2$

[From [Mar13]]

## Case $s \neq 0$

$$P = \begin{bmatrix} f_x r_{11} + s r_{21} + u_0 r_{31} & f_x r_{12} + s r_{22} + u_0 r_{32} & f_x r_{13} + s r_{23} + u_0 r_{33} & f_x t_1 + s t_2 + u_0 t_3 \\ f_y r_{21} + v_0 r_{31} & f_y r_{22} + v_0 r_{32} & f_y r_{23} + v_0 r_{33} & f_y t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

$$u_0 = \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix}$$

$$f_x = \left\| \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix} - u_0 \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} \right\|$$

$$v_0 = \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix}$$

$$\begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix} = \begin{bmatrix} p_{21} \\ p_{22} \\ p_{23} \end{bmatrix} - v_0 \cdot \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \end{bmatrix} \cdot \frac{1}{f_y}$$

and we still have  $r_3$  and  $t_3$ .

[From [Mar13]]

## Case $s \neq 0$

$$P = \begin{bmatrix} f_x r_{11} + s r_{21} + u_0 r_{31} & f_x r_{12} + s r_{22} + u_0 r_{32} & f_x r_{13} + s r_{23} + u_0 r_{33} & f_x t_1 + s t_2 + u_0 t_3 \\ f_y r_{21} + v_0 r_{31} & f_y r_{22} + v_0 r_{32} & f_y r_{23} + v_0 r_{33} & f_y t_2 + v_0 t_3 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

$$r_2 = \frac{r_2}{\|r_2\|}$$

$$r_1 = r_2 \times r_3$$

$$t_2 = \frac{p_{24} - v_0 \cdot t_3}{f_y}$$

$$t_1 = \frac{p_{14} - u_0 \cdot t_3 - s \cdot t_2}{f_x}$$

[From [Mar13]]

$$\begin{bmatrix} f_x \\ s \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix}^{-1} \cdot \begin{bmatrix} p_{11} - u_0 r_{31} \\ p_{12} - u_0 r_{32} \end{bmatrix}$$

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## Non linear approach principle

Consider the projection of a 3D point  $X_w = \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}$  onto a 2D point

$\mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$  with  $z_i = f$  (the focal length):

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \lambda_i \left[ \mathbf{R} \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} + \mathbf{t} \right]$$

with :

- $\mathbf{R}$  the rotation matrix
- $\mathbf{t}$  the translation vector
- $\lambda_i$  a scale factor

## Non linear approach principle

Suppressing the scale factor, we can write:

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} f \frac{r_{11}x_w + r_{12}y_w + r_{13}z_w + t_1}{r_{31}x_w + r_{32}y_w + r_{33}z_w + t_3} \\ f \frac{r_{21}x_w + r_{22}y_w + r_{23}z_w + t_2}{r_{31}x_w + r_{32}y_w + r_{33}z_w + t_3} \end{bmatrix}$$

Expressing  $\begin{bmatrix} x_i \\ y_i \end{bmatrix}$  in pixels coordinates:

$$\begin{aligned} x_i &= \frac{(u + e_x - u_0)}{s_x} - d_{0_x} \\ y_i &= \frac{(v + e_y - v_0)}{s_y} - d_{0_y} \end{aligned}$$

- $e_x, e_y$ : measurement errors on  $x_i, y_i$
- $s_x, s_y$ : imager size along  $x$  and  $y$
- $d_{0_x}, d_{0_y}$ : distortions

[From [Mar13]]

## Non linear approach principle

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} f \frac{r_{11}x_w + r_{12}y_w + r_{13}z_w + t_1}{r_{31}x_w + r_{32}y_w + r_{33}z_w + t_3} \\ f \frac{r_{21}x_w + r_{22}y_w + r_{23}z_w + t_2}{r_{31}x_w + r_{32}y_w + r_{33}z_w + t_3} \end{bmatrix} = \begin{bmatrix} \frac{(u + e_x - u_0)}{s_x} - d_{0_x} \\ \frac{(v + e_y - v_0)}{s_y} - d_{0_y} \end{bmatrix}$$

We can express:

$$\begin{bmatrix} u + e_x \\ v + e_y \end{bmatrix} = \begin{bmatrix} P(\Phi) \\ Q(\Phi) \end{bmatrix}$$

with  $\Phi = [u_0, v_0, k_1, k_2, k_3, p_1, p_2, f_x, f_y, t_1, t_2, t_3, \alpha, \beta, \gamma]^T$  a 15 parameters vector where  $(\alpha, \beta, \gamma)^T$  parametrize the orientation.

[From [Mar13]]

## Non linear approach principle

Then:  $\begin{bmatrix} e_x \\ e_y \end{bmatrix} = \begin{bmatrix} P(\Phi) - u \\ Q(\Phi) - v \end{bmatrix} \Rightarrow V(\Phi) = \begin{bmatrix} e_x \\ e_y \end{bmatrix}$

We must find  $\Phi$  which minimizes the reprojection error  $S$ :

- if 1 image and  $n$  correspondence points:

$$S = \sum_{i=1}^n (e_{x_i}^2 + e_{y_i}^2)$$

- if  $m$  image and  $n$  correspondence points on each image:

$$S = \sum_{j=1}^m \sum_{i=1}^n (e_{x_{ij}}^2 + e_{y_{ij}}^2)$$

[From [Mar13]]

## Non linear approach principle

Then:  $\begin{bmatrix} e_x \\ e_y \end{bmatrix} = \begin{bmatrix} P(\Phi) - u \\ Q(\Phi) - v \end{bmatrix} \Rightarrow V(\Phi) = \begin{bmatrix} e_x \\ e_y \end{bmatrix}$

We must find  $\Phi$  which minimizes the reprojection error  $S$ :

- if 1 image and  $n$  correspondence points:  $S = \sum_{i=1}^n (e_{x_i}^2 + e_{y_i}^2)$
- if  $m$  image and  $n$  correspondence points on each image:  
 $S = \sum_{j=1}^m \sum_{i=1}^n (e_{x_{ij}}^2 + e_{y_{ij}}^2)$

Non-linear optimization problem!

That can be solved using algorithm as Gauss-Newton or Levenberg-Marquardt.

[From [Mar13]]

## Non linear approach principle

Then:  $\begin{bmatrix} e_x \\ e_y \end{bmatrix} = \begin{bmatrix} P(\Phi) - u \\ Q(\Phi) - v \end{bmatrix} \Rightarrow V(\Phi) = \begin{bmatrix} e_x \\ e_y \end{bmatrix}$

We must find  $\Phi$  which minimizes the reprojection error  $S$ :

Note: An implementation for learning purposes of a pose estimation using Gauss-Newton can be found at

► <http://people.rennes.inria.fr/Eric.Marchand/pose-estimation/tutorial-pose-gauss-newton-opencv.html>

[From [Mar13]]

## Outline

### 1 Camera model and calibration

- Pinhole Camera Model
- Calibration parameters
- Calibration methods
  - Principle
  - A linear method: Direct Linear Transform (DLT)
  - From  $\mathbf{P}$  to  $\mathbf{K}$ ,  $\mathbf{R}$  and  $\mathbf{t}$
  - Non linear approach principle
  - "Gold standard" algorithm
- References, tools and demo

### 2 Some pose estimation algorithms with a known 3D model

### 3 Transformation between images

### 4 Motion estimation

### 5 Conclusion

### 6 Links and bibliography



## "Gold standard" algorithm [HZ04]

### Objective

Given  $n \geq 6$  world to image point correspondences  $X_w \leftrightarrow \mathbf{x}_i$ , determine the Maximum Likelihood estimated of the camera projection matrix  $\mathbf{P}$ , i.e. the  $\mathbf{P}$  which minimizes  $\sum_i d(\mathbf{x}_i, \mathbf{P}X_w)$ .  $\sum_i d(\mathbf{x}_i, \mathbf{P}X_w)$  is the geometric error in the image. Minimizing geometric error require the use of iterative techniques (as Levenberg-Marquardt). If the measurement error are Gaussian then the solution of

$$\min_{\mathbf{P}} \sum_i d(\mathbf{x}_i, \mathbf{P}X_w)$$

is the Maximum Likelihood estimate under  $\mathbf{P}$ .

DLT solution (or a minimal solution) is used as a starting point for the iterative minimization.

[From [HZ04]

## "Gold standard" algorithm [HZ04]

### Objective

Given  $n \geq 6$  world to image point correspondences  $X_w \leftrightarrow \mathbf{x}_i$ , determine the Maximum Likelihood estimated of the camera projection matrix  $\mathbf{P}$ , i.e. the  $\mathbf{P}$  which minimizes  $\sum_i d(\mathbf{x}_i, \mathbf{P}X_w)$

- 1 **Linear Solution** Compute an initial estimate of  $\mathbf{P}$  using previous linear method.
  - 1 **Normalization** Use a similarity transformation  $T$  to normalize the image points  $\tilde{\mathbf{x}}_i = T\mathbf{x}_i$ , and a second similarity transformation  $U$  to normalize the space points  $\tilde{X}_w = UX_w$
  - 2 Apply **DLT** algorithm
- 2 **Minimize geometric error** Using the linear estimate as a starting point minimize the geometric error

$$\sum_i d(\tilde{\mathbf{x}}_i, \tilde{\mathbf{P}}\tilde{X}_w)$$

over  $\tilde{\mathbf{P}}$ , using an iterative algorithm such as Levenberg-Marquardt.

## "Gold standard" algorithm [HZ04]

### Objective

Given  $n \geq 6$  world to image point correspondences  $X_w \leftrightarrow \mathbf{x}_i$ , determine the Maximum Likelihood estimated of the camera projection matrix  $\mathbf{P}$ , i.e. the  $\mathbf{P}$  which minimizes  $\sum_i d(\mathbf{x}_i, \mathbf{P}X_w)$

- 1 **Linear Solution** Compute an initial estimate of  $\mathbf{P}$  using previous linear method.
- 2 **Minimize geometric error** Using the linear estimate as a starting point minimize the geometric error

$$\sum_i d(\tilde{\mathbf{x}}_i, \tilde{\mathbf{P}}\tilde{X}_w)$$

over  $\tilde{\mathbf{P}}$ , using an iterative algorithm such as Levenberg-Marquardt.

- 3 **Denormalization** The camera matrix for the original (unnormalized) coordinates is obtained from  $\tilde{\mathbf{P}}$  as

$$\mathbf{P} = T^{-1}\tilde{\mathbf{P}}U$$

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## References

Presented methods were proposed by Roberts [Rob63], Tsai [Tsa87], Lowe [Low85, Low91], Yuan [Yua89] and Zhang [Zha00] (among others).

## Tools

- OpenCV <https://opencv.org> (*Opensource*)
- OpenGV <https://laurentkneip.github.io/opengv/index.html>
- ViSP <https://visp.inria.fr> (*Opensource*)
- MATLAB Toolboxes <https://fr.mathworks.com/solutions/image-video-processing.html> or <http://www.vision.caltech.edu/bouguetj/calib.doc/>
- Omnidirectional Calibration Toolbox <http://www.robots.ox.ac.uk/~cmei/Toolbox.html#download>

Pose estimation for augmented reality: a hand-on survey [MUS16]  
codes and explanations <http://people.rennes.inria.fr/Eric.Marchand/pose-estimation/index.html>

## Calibration with a chessboard

Any appropriately characterized object could be used as a calibration object

⇒ practically: use of regular pattern as a chessboard

Rq.: The specific use of this calibration object and much of the calibration approach itself comes from [Zha00] and [SM99]

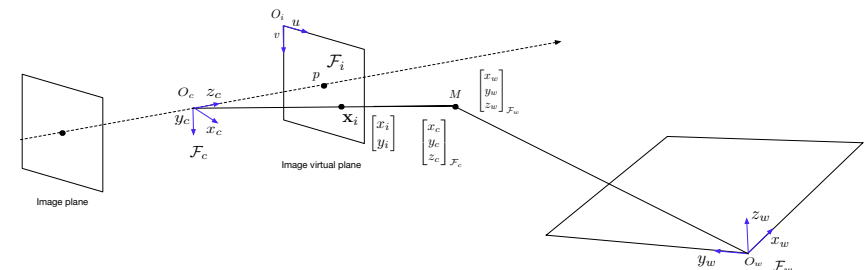
Demo: openCV calibration example code

<https://github.com/opencv/opencv/blob/master/samples/cpp/calibration.cpp>

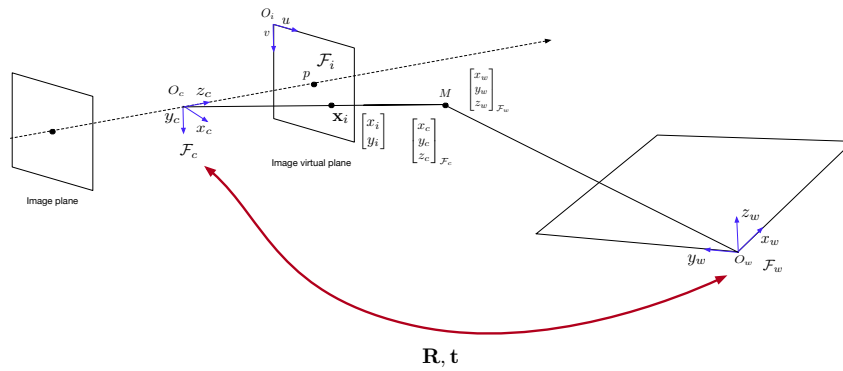
In the yml file, you will find the camera matrix

$\begin{bmatrix} f_x, 0, u_0, 0, f_y, v_0, 0, 0, 1 \end{bmatrix}$  and the distortion coefficients  
 $\begin{bmatrix} k_1, k_2, p_1, p_2, k_3 \end{bmatrix}$

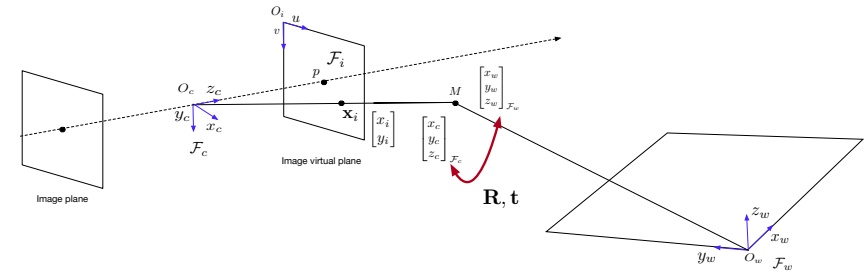
## Summary on perspective projection



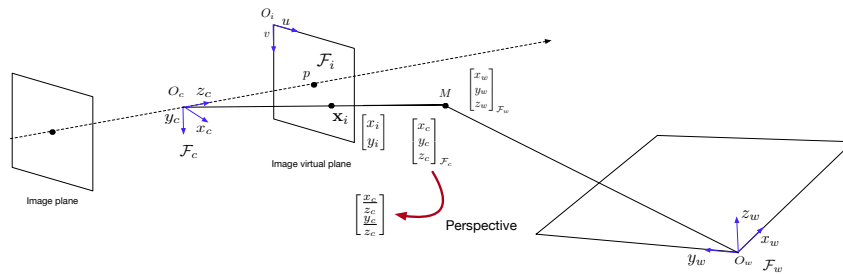
## Summary on perspective projection



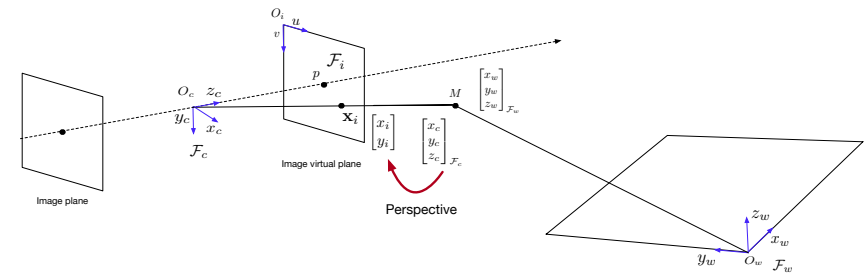
## Summary on perspective projection



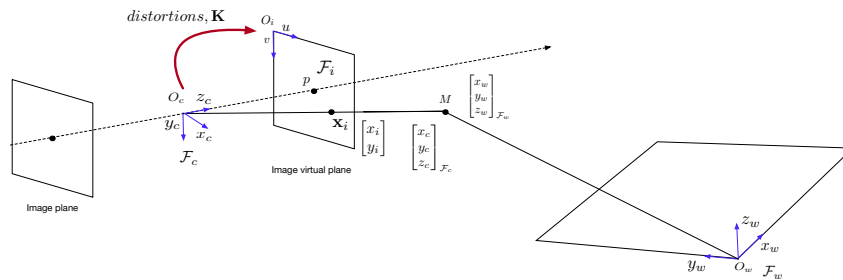
## Summary on perspective projection



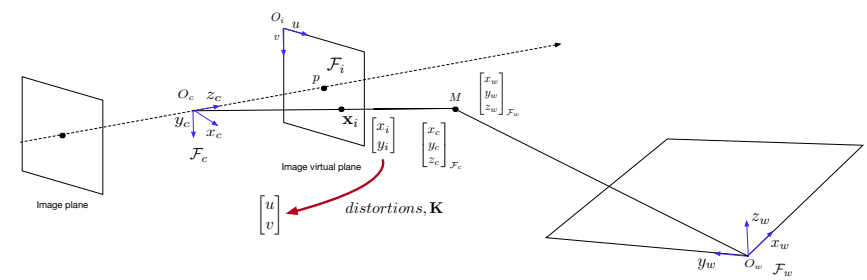
## Summary on perspective projection



## Summary on perspective projection



## Summary on perspective projection



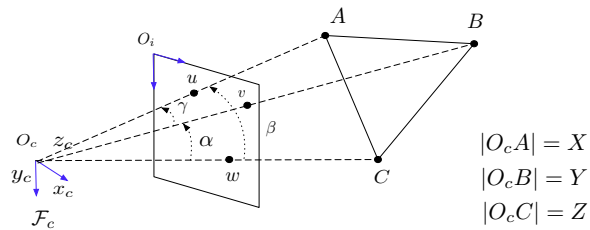
## Outline

- 1 Camera model and calibration
- 2 Some pose estimation algorithms with a known 3D model
  - P3P
  - POS and POSIT [DD95]
  - EPnP algorithm [LFNP09]
  - Others EPnP algorithms
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## P3P : pose estimation with the smallest correspondence subset



Most of the P3P approaches rely on a 2 steps solution:

- Estimation of the unknown depth of each point (in  $\mathcal{F}_c$ )
- Estimating the rigid transformation that maps the coordinates expressed in  $\mathcal{F}_c$  to the coordinates expressed in  $\mathcal{F}_w$

## P3P : pose estimation with the smallest correspondence subset

- Estimation of the unknown depth of each point (in  $\mathcal{F}_c$ ):
  - Use of triangles  $O_cAB$ ,  $O_cBC$  and  $O_cAC$  and the law of cosines:

$$Y^2 + Z^2 - 2YZ \cos \alpha - |BC|^2 = 0$$

$$Z^2 + X^2 - 2ZX \cos \beta - |AC|^2 = 0$$

$$X^2 + Y^2 - 2XY \cos \gamma - |AB|^2 = 0$$

- Solve a fourth order polynomial equation (example in [GHTC03])
  - up to 4 four possible solutions

- Estimating the rigid transformation from  $\mathcal{F}_c$  to  $\mathcal{F}_w$

Necessity to have a 4th point to disambiguate the results.

## Outline

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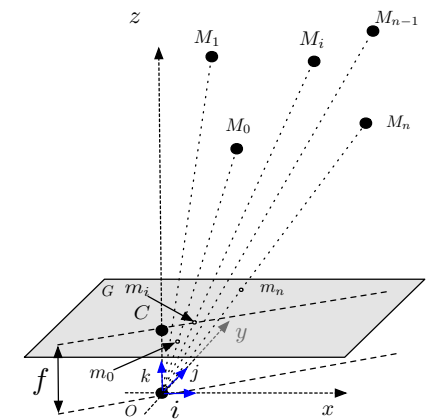
## POS and POSIT [DD95]: overview of the problem

Notations:

- $O$  center of projection
- $(\vec{i}, \vec{j}, \vec{k})$  the camera frame
- $f$  focal length
- $G$  the image plane
- $C$  the central point

Hypothesis:

- pinhole camera model
- $n$  non coplanar 3D points:  $M_0, M_1, \dots, M_n$
- $m_0, m_1, \dots, m_n$ : perspective projection in image plane of the corresponding 3D points



## POS and POSIT [DD95]: overview of the problem

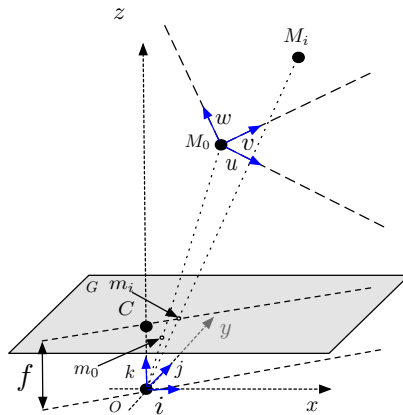
Notations:

- $\mathcal{F}_o$ : object frame as  $M_0$  origin with  $(u, v, w)$
- $M_i = \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix}$  in  $\mathcal{F}_o$
- $m_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$  in the image plane

Hypothesis:

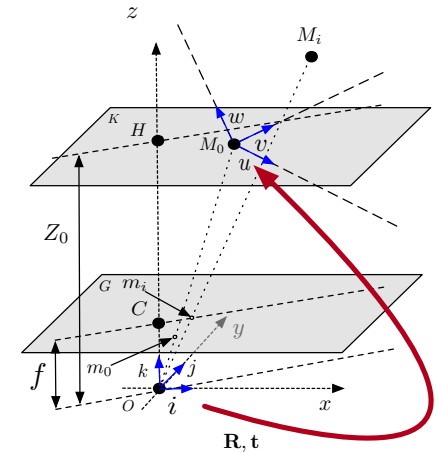
- the shape of the object is known  $\Rightarrow M_i$  coordinates in  $\mathcal{F}_o$  are known

We want to retrieve  $[X_i, Y_i, Z_i]^T$  coordinates of  $M_i$  in  $\mathcal{F}_c$



## POS and POSIT [DD95]: overview of the problem

- $K$ : plane parallel to image plane  $G$  containing  $M_0$  at  $Z_0$  from  $\mathcal{F}_c$
- Problem: compute  $\mathbf{R}, \mathbf{t}$

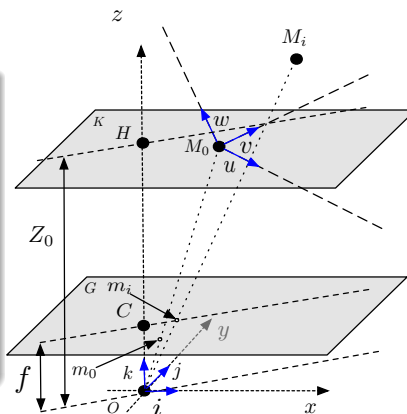


## POS and POSIT [DD95]: Compute $\mathbf{R}, \mathbf{t}$

**R**

$$R = \begin{bmatrix} i_u & i_v & i_w \\ j_u & j_v & j_w \\ k_u & k_v & k_w \end{bmatrix} = \begin{bmatrix} \vec{i}^T \\ \vec{j}^T \\ \vec{k}^T \end{bmatrix}$$

only need to compute  $\vec{i}, \vec{j}$  then  $\vec{k} = \vec{i} \times \vec{j}$



## POS and POSIT [DD95]: Compute $\mathbf{R}, \mathbf{t}$

**t**

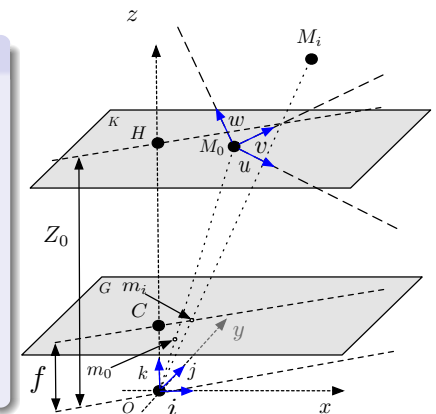
Given by  $O\vec{M}_0$ :

$$\mathbf{t} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix}$$

and  $O, m_0$  and  $M_0$  aligned:

$$\mathbf{t} = \frac{Z_0}{f} O\vec{m}_0$$

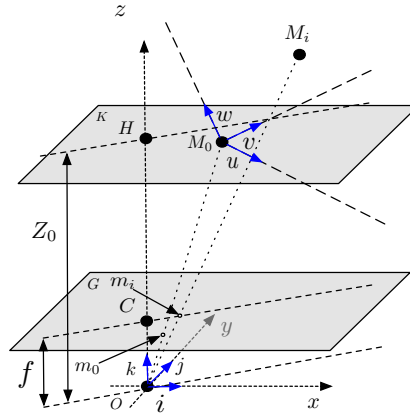
Then to get  $\mathbf{t}$ , only  $Z_0$  is missing.





## POS and POSIT [DD95]: Compute $\mathbf{R}, \mathbf{t}$

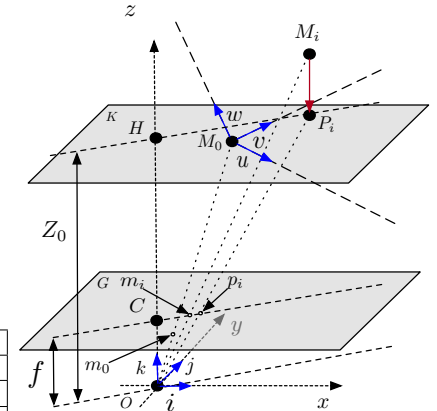
The object pose is fully defined once we have  $\vec{i}, \vec{j}$  and  $Z_0$ .



## POS and POSIT [DD95]: Compute $\mathbf{R}, \mathbf{t}$

- Scaled orthographic projection (SOP):
  - ▶ an approximation to *true* perspective projection
  - ▶ consider an object that we can consider all points with the same depth
- SOP vs. Perspective projection (PP):

SOP	PP
$M_i \Rightarrow P_i \Rightarrow p_i$	$M_i \Rightarrow m_i$
$p_i = (x'_i, y'_i)^T$	$m_i = (x_i, y_i)^T$
$p_i = (f \frac{X_i}{Z_0}, f \frac{Y_i}{Z_0})^T$	$m_i = (f \frac{X_i}{Z_i}, f \frac{Y_i}{Z_i})^T$



## POS and POSIT [DD95]: Compute $\mathbf{R}, \mathbf{t}$

- SOP *scaling factor*:  $s = \frac{f}{Z_0}$
- $M_0$  projects on  $m_0$  with SOP and PP

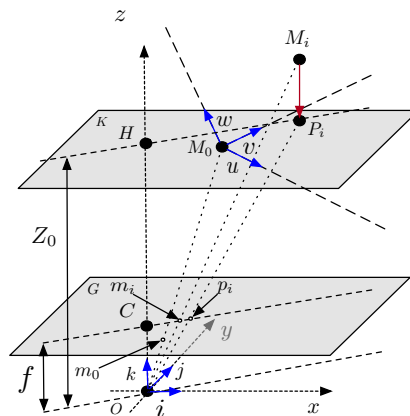
We can write:

$$\begin{aligned} x'_i &= f \frac{X_i}{Z_0} = f \frac{X_i - X_0 + X_0}{Z_0} \\ &= x_0 + s(X_i - X_0) \end{aligned}$$

And deduce:

$$\begin{aligned} x'_i &= x_0 + s(X_i - X_0) \\ y'_i &= y_0 + s(Y_i - Y_0) \end{aligned}$$

$$\text{and: } \vec{m_0} p_i = \frac{f}{Z_0} \vec{M_0} P_i$$



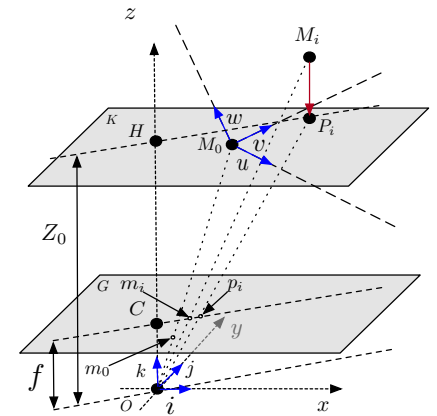
## POS and POSIT [DD95]: Compute $\mathbf{R}, \mathbf{t}$

- Known coordinates of  $\vec{M_0} \vec{M_i}$  and  $\vec{m_0} \vec{m_i}$
- Unknown:  $\vec{i}, \vec{j}$  and  $Z_0$
- Links between known and unknown:

$$\vec{M_0} \vec{M_i} \cdot \frac{f}{Z_0} \vec{i} = x_i(1 + \epsilon_i) - x_0$$

$$\vec{M_0} \vec{M_i} \cdot \frac{f}{Z_0} \vec{j} = y_i(1 + \epsilon_i) - y_0$$

$$\text{with } \epsilon_i = \frac{1}{Z_0} \vec{M_0} \vec{M_i} \cdot \vec{k} \quad [\text{demonstration in [DD95]}]$$



## POS and POSIT [DD95]: Compute $\mathbf{R}, \mathbf{t}$

Re-writing:

$$M_0 \vec{M}_i \cdot \vec{I} = x_i(1 + \epsilon_i) - x_0$$

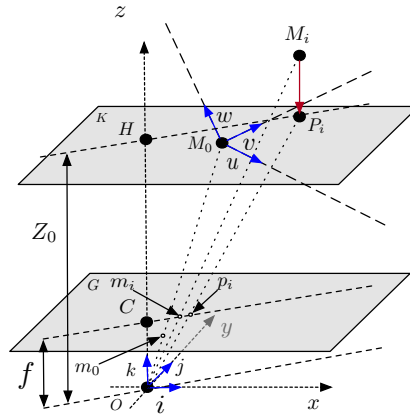
$$M_0 \vec{M}_i \cdot \vec{J} = y_i(1 + \epsilon_i) - y_0$$

with

$$\vec{I} = \frac{f}{Z_0} \vec{i}$$

$$\vec{J} = \frac{f}{Z_0} \vec{j}$$

So if give values to  $\epsilon_i \Rightarrow$  linear system with two unknowns  $\vec{I}$  and  $\vec{J}$



## POS and POSIT [DD95]: POS algorithm

POS algorithm: *Pose from Orthography and Scaling*

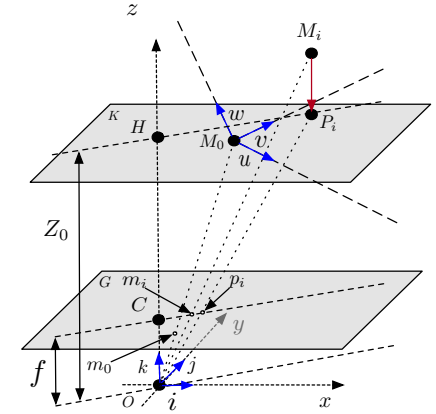
- Give an approximate value to  $\epsilon_i$
- Solve linear system with unknowns  $\vec{I}$  and  $\vec{J}$
- Get  $\vec{i}, \vec{j}, \vec{k}$  with

$$\vec{i} = \vec{I} / \|\vec{I}\|$$

$$\vec{j} = \vec{J} / \|\vec{J}\|$$

$$\vec{k} = \vec{i} \times \vec{j}$$

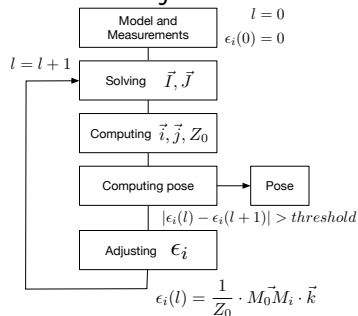
- Get  $Z_0 = \frac{f}{\vec{i}}$



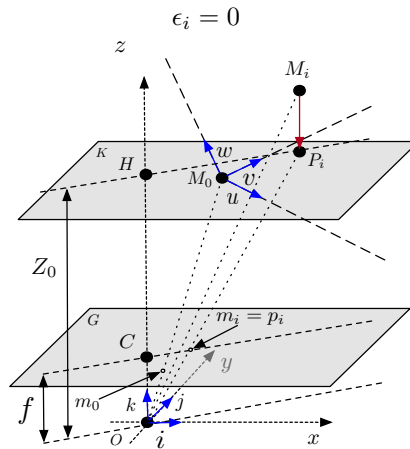
## POS and POSIT [DD95]: POSIT algorithm

POSIT algorithm: POS with iterations

- Solve  $\vec{I}$  and  $\vec{J}$  then estimate  $\epsilon_i$



[From [Mar13]]



## POS and POSIT [DD95]: POSIT algorithm

- Extended to coplanar feature points in [ODD96]
- Note: an implementation of POSIT for leaning purposes can be found at

• <http://people.rennes.inria.fr/Eric.Marchand/pose-estimation/tutorial-pose-dementhon-opencv.html>

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## EPnP algorithm [LFNP09]

- non-iterative solution to the PnP problem applicable for all  $n \geq 4$
- handles both planar and non-planar configurations
- $n$  3D points coordinates expressed as a weighted sum of four *virtual* control points
- pose problem : estimation of the coordinates of control points in  $\mathcal{F}_c$
- can be done in  $O(n)$ 
  - ▶ expressing these coordinates as weighted sum of the eigenvectors of a  $12 \times 12$  matrix
  - ▶ solving a small constant number of quadratic equations to pick the right weights

## EPnP algorithm [LFNP09]: Parameterization in the General Case

- $p_i, i = 1, \dots, n$ : the  $n$  points whose 3D coordinates are known in  $\mathcal{F}_w$
- $c_j, j = 1, \dots, 4$ : the 4 control points coordinates in  $\mathcal{F}_w$
- 

$$p_i^w = \sum_{j=1}^4 \alpha_{ij} c_j^w, \text{ with } \sum_{j=1}^4 \alpha_{ij} = 1$$

where the  $\alpha_{ij}$  are homogeneous barycentric coordinates.

- Can also be expressed in  $\mathcal{F}_c$ :  $p_i^c = \sum_{j=1}^4 \alpha_{ij} c_j^c$
- In theory the control points can be chosen arbitrarily, but for stability reason:
  - ▶ taking the centroid of the  $n$  reference points as one
  - ▶ select the others in such a way that they form a basis aligned with the principal directions

## EPnP algorithm [LFNP09]: The Solution as Weighted Sum of Eigenvectors

- $x_{i=1,\dots,n}$  the 2D projections of the  $p_{i=1,\dots,n}$  reference points:

$$\forall i, \lambda_i \begin{bmatrix} x_i \\ 1 \end{bmatrix} = \mathbf{K} p_i^c = \mathbf{K} \sum_{j=1}^4 \alpha_{ij} c_j^c$$

- with  $c_j^c = [x_j^c, y_j^c, z_j^c]^T$  and  $x_i = [u_i, v_i]^T$ :

$$\forall i, \lambda_i \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix} \sum_{j=1}^4 \alpha_{ij} \begin{bmatrix} x_j^c \\ y_j^c \\ z_j^c \end{bmatrix}$$

- ▶ The unknown parameters of this linear system are the 12 control point coordinates  $\{(x_j^c, y_j^c, z_j^c)\}_{j=1,\dots,4}$  and the  $n$  projective parameters  $\{\lambda_i\}_{i=1,\dots,n}$ .
- ▶ last row implies  $\lambda_i = \sum_{j=1}^4 \alpha_{ij} z_j^c$

## EPnP algorithm [LFNP09]: The Solution as Weighted Sum of Eigenvectors

- Substituting  $\lambda_i$  expression in the first two rows gives two linear equations for each reference point:

$$\sum_{j=1}^4 \alpha_{ij} f_x x_j^c + \alpha_{ij} (u_0 - u_i) z_j^c = 0$$

$$\sum_{j=1}^4 \alpha_{ij} f_y y_j^c + \alpha_{ij} (v_0 - v_i) z_j^c = 0$$

- $\lambda_i$  does not appear anymore in those equations
- concatenating them for all  $n$  reference points give a linear system:  $Mx = 0$  where
  - $x = [c_1^c, c_2^c, c_3^c, c_4^c]^T$  is a 12-vector made of the unknowns
  - $M$  is a  $n \times 12$  matrix generated by arranging the coefficients of the two last equations for each reference point

## EPnP algorithm [LFNP09]: The Solution as Weighted Sum of Eigenvectors

Solving  $Mx = 0$

- the solution therefore belongs to the null space of  $M$ , and can be expressed as

$$x = \sum_{i=1}^N \beta_i \mathbf{v}_i$$

where the set  $\mathbf{v}_i$  are the columns of the right-singular vectors of  $M$  corresponding to the  $N$  null singular values of  $M$

- they can be computed as the null eigenvectors of matrix  $M^T M$ 
  - $M^T M$  is  $12 \times 12$
  - Computing  $M^T M$  has  $O(n)$  complexity

## EPnP algorithm [LFNP09]: Choosing the Right Linear Combination

- the effective dimension  $N$  of the null space of  $M^T M$  can vary from 1 to 4 depending on the configuration of the reference points, the focal length of the camera, and the amount of noise
- compute solutions for all four values of  $N$  and keep the one that yields the smallest reprojection error

$$res = \sum_i d^2 \left( K[R|t] \begin{pmatrix} p_i^w \\ 1 \end{pmatrix}, x_i \right)$$

with  $d(\tilde{a}, b)$  the 2D distance between point  $a$  expressed in homogeneous coordinates, and point  $b$ .

Note: Details on the 4 cases are given in [LFNP09]

## EPnP algorithm [LFNP09]

- non-iterative solution to the  $PnP$  problem applicable for all  $n \geq 4$
- handles both planar and non-planar configurations
- $n$  3D points coordinates expressed as a weighted sum of four *virtual* control points
- pose problem : estimation of the coordinates of control points in  $\mathcal{F}_c$
- To improve accuracy: the output of the closed-form solution can be used to initialize a Gauss-Newton scheme that will choose the values  $\beta_i$  that minimize the change in distance between control points

## Outline

- 1 Camera model and calibration
- 2 Some pose estimation algorithms with a known 3D model
  - P3P
  - POS and POSIT [DD95]
  - EPnP algorithm [LFNP09]
  - Others EPnP algorithms
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## Others EPnP algorithms

- EPnP[LFNP09]: first accurate  $O(N)$  solution to the PnP
- Other interesting  $O(n)$  solutions when the number of point correspondences increases:
  - ▶ OPnP [ZKS<sup>+</sup>13]: Optimal PnP – parameterize the rotation by using non-unit quaternion and formulate the PnP problem into an unconstrained optimization problem.
  - ▶ GPnP [KFS13]: non-iterative n-point solution with linear complexity in the number of points – Extension of EPnP to Non-Perspective-n-Point problem (NPnP problem)
  - ▶ UPnP [KLS14]: Universal PnP – applicable to both central and non-central camera systems
  - ▶ MLPnP [ULH16]: real-time Maximum Likelihood solution to the Perspective-n-Point problem – statistically optimal solution to PnP
  - ▶ ...

## Others EPnP algorithms

- It also exists iterative techniques as for ex. LHM [LHM00]: orthogonal iteration method to directly minimize the object space error
- A good review as speed and accuracy of 13 PnP methods can be found in [ULH16]  
⇒ MLPnP has similar execution times compared to the fastest methods (EPnP still faster) and is better in terms of accuracy.
- Algorithms implementations can be found in OpenCV and OpenGV [KLS14]

## PnP calculation demo

- Code from OpenCV real time pose estimation of a textured object tutorial  
[https://docs.opencv.org/3.3.0/dc/d2c/tutorial\\_real\\_time\\_pose.html](https://docs.opencv.org/3.3.0/dc/d2c/tutorial_real_time_pose.html)
- Estimate the camera pose in order to track a textured object with six degrees of freedom given a 2D image and its 3D textured model
  - ▶ Read 3D textured object model and object mesh
  - ▶ Take input from Camera or Video
  - ▶ Extract ORB features and descriptors from the scene
  - ▶ Match scene descriptors with model descriptors
  - ▶ Pose estimation using PnP + Ransac
  - ▶ Linear Kalman Filter for bad poses rejection

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  - How to get correspondences ?
  - 3D reconstruction

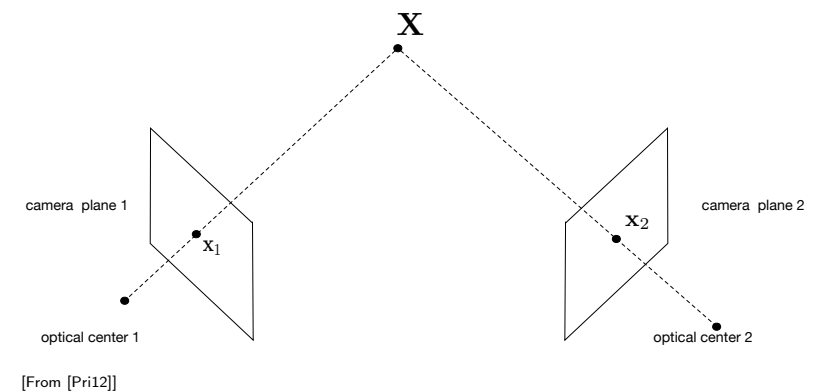
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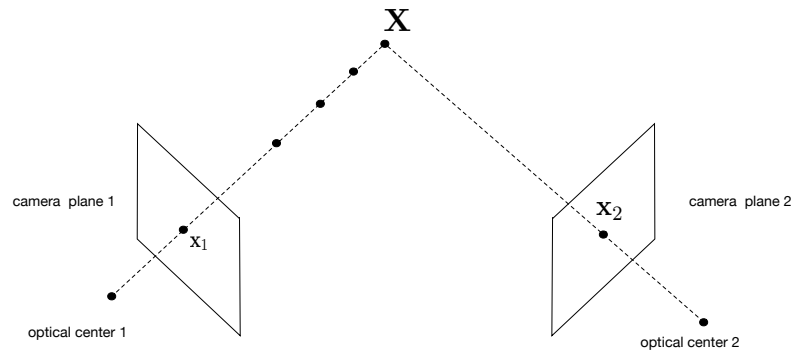
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## The epipolar constraint

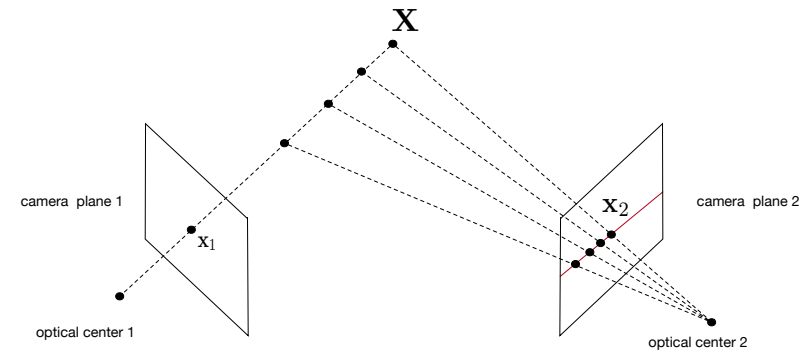


## The epipolar constraint



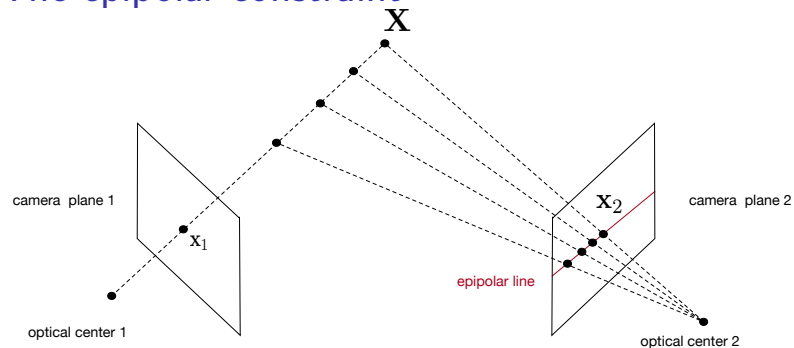
[From [Pri12]]

## The epipolar constraint



[From [Pri12]]

## The epipolar constraint



- *epipolar constraint*: for any point in the first image, the corresponding point in the second image is constrained to lie on a line
- The epipolar line depends on the intrinsic and extrinsic parameters of the cameras

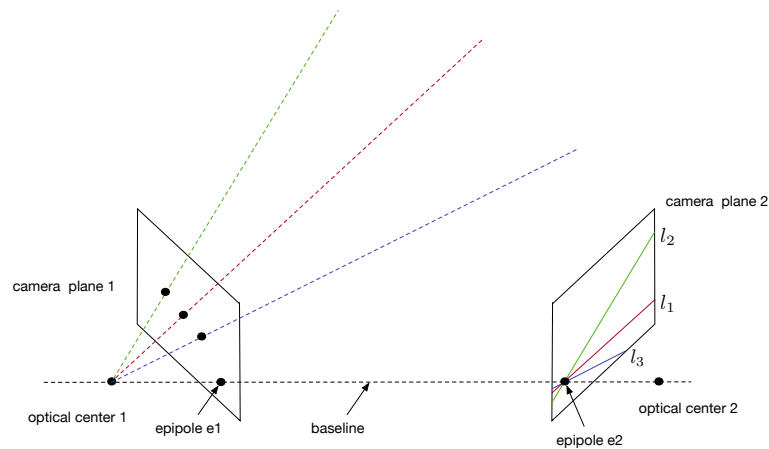
[From [Pri12]]

## The epipolar constraint

Practical applications [Pri12]:

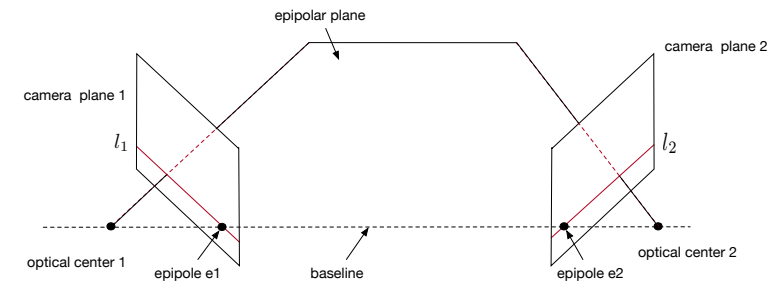
- Finding point correspondences (given intrinsic and extrinsic parameters): for a point in the first image, perform a 1D search along the epipolar line in the second image for the corresponding position
- Constraint on corresponding points is a function of the intrinsic and extrinsic parameters
  - ⇒ Use the observed pattern of point correspondences to determine the extrinsic parameters
  - ⇒ Get the geometric relationship between the two cameras

## The epipole



[From [Pri12]]

## The epipolar plane



[From [Sze10]]

## Outline

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## Normalized coordinates [HZ04]

Let's consider:

- $\tilde{X}_w = \begin{bmatrix} X_w \\ 1 \end{bmatrix}$  a point in the world in homogeneous coordinates
- Camera matrix  $\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$  and  $\lambda \tilde{\mathbf{x}}_i = \mathbf{P} \tilde{X}_w$  a point in the image
- $\mathbf{K}$  is known

We can get normalized coordinates:

- $\lambda \tilde{\mathbf{x}}_i = \lambda \mathbf{K}^{-1} \tilde{X}_i \Rightarrow \lambda \tilde{\mathbf{x}}_i = [\mathbf{R}|\mathbf{t}] \tilde{X}_w$  ( $\tilde{\mathbf{x}}_i$  stills in homogeneous coordinates)
- equivalent to a camera where  $\mathbf{K} = \mathbf{I}$

Normalized camera matrix:  $\mathbf{P}' = \mathbf{K}^{-1}\mathbf{P} = [\mathbf{R}|\mathbf{t}]$



## Essential matrix [Pri12]

The geometric relationship between the two cameras is captured by the essential matrix.

- Assume normalized cameras, first camera at origin.

$$\lambda_1 \hat{\mathbf{x}}_{i1} = [\mathbf{I} | 0] \tilde{\mathbf{X}}_w$$

$$\lambda_2 \hat{\mathbf{x}}_{i2} = [\mathbf{R} | \mathbf{t}] \tilde{\mathbf{X}}_w$$

- 1<sup>st</sup> camera:  $\lambda_1 \hat{\mathbf{x}}_{i1} = \mathbf{X}_w$
- 2<sup>nd</sup> camera:  $\lambda_2 \hat{\mathbf{x}}_{i2} = \mathbf{R} \mathbf{X}_w + \mathbf{t}$
- Substituting:

$$\lambda_2 \hat{\mathbf{x}}_{i2} = \lambda_1 \mathbf{R} \hat{\mathbf{x}}_{i1} + \mathbf{t}$$

## Essential matrix [Pri12]

- Constraint between the possible positions of corresponding points in the two images

$$\lambda_2 \hat{\mathbf{x}}_{i2} = \lambda_1 \mathbf{R} \hat{\mathbf{x}}_{i1} + \mathbf{t}$$

- take cross product with  $\mathbf{t}$ :

$$\lambda_2 \mathbf{t} \times \hat{\mathbf{x}}_{i2} = \lambda_1 \mathbf{t} \times \mathbf{R} \hat{\mathbf{x}}_{i1}$$

- take inner product with  $\hat{\mathbf{x}}_{i2}$ :

$$\hat{\mathbf{x}}_{i2}^T \mathbf{t} \times \mathbf{R} \hat{\mathbf{x}}_{i1} = 0$$

## Essential matrix [Pri12]

$$\hat{\mathbf{x}}_{i2}^T \mathbf{t} \times \mathbf{R} \hat{\mathbf{x}}_{i1} = 0$$

- The cross product term can be expressed as a matrix:

$$\mathbf{t}_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

- Defining the *essential matrix*:

$$\mathbf{E} = \mathbf{t}_\times \mathbf{R}$$

- and the essential matrix relation:

$$\hat{\mathbf{x}}_{i2}^T \mathbf{E} \hat{\mathbf{x}}_{i1} = 0$$

## Properties of the essential matrix [Pri12]

$$\hat{\mathbf{x}}_{i2}^T \mathbf{E} \hat{\mathbf{x}}_{i1} = 0$$

- Rank 2:  $\det[\mathbf{E}] = 0$
- 5 D.o.F.
- Non-linear constraint between elements:

$$2\mathbf{E}\mathbf{E}^T\mathbf{E} - \text{trace}[\mathbf{E}\mathbf{E}^T]\mathbf{E} = 0$$

## Computing the essential matrix

- 5-point algorithm [Nis04]
- 8-point algorithm [Lh81]
  - ▶  $\hat{\mathbf{x}}_{i2}^T \mathbf{E} \hat{\mathbf{x}}_{i1} = 0$
  - ▶ can be solved with SVD

## Recovering epipolar lines [Pri12]

- Equation of a line:

$$ax + by + c = 0$$

- or:

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

- or:

$$l\tilde{x} = 0$$

## Recovering epipolar lines [Pri12]

- Equation of a line :  $l\tilde{x} = 0$

- Now consider:

$$\hat{\mathbf{x}}_{i2}^T \mathbf{E} \hat{\mathbf{x}}_{i1} = 0$$

- This as the form  $l_1 \hat{\mathbf{x}}_{i1} = 0$  where  $l_1 = \hat{\mathbf{x}}_{i2}^T \mathbf{E}$
- So the epipolar lines can be expressed as:

$$l_1 = \hat{\mathbf{x}}_{i2}^T \mathbf{E}$$

$$l_2 = \hat{\mathbf{x}}_{i1}^T \mathbf{E}$$

## Recovering epipoles [Pri12]

- Every epipolar line in image 1 passes through the epipole  $e_1$

$$\Rightarrow \hat{\mathbf{x}}_{i2}^T \mathbf{E} \tilde{e}_1 = 0$$

for all  $\hat{\mathbf{x}}_{i2}^T$

- This can only be true if  $\tilde{e}_1$  is in the nullspace of  $\mathbf{E}$ :

$$\tilde{e}_1 = \text{null}[\mathbf{E}]$$

- Similarly:

$$\tilde{e}_2 = \text{null}[\mathbf{E}^T]$$

We find the null spaces by computing the SVD of  $\mathbf{E}$ :

$$\mathbf{E} = \mathbf{U} \mathbf{D} \mathbf{V}^T$$

and taking  $\tilde{e}_1$  the last column of  $\mathbf{V}$  and  $\tilde{e}_2$  the last row of  $\mathbf{U}$

## Retrieving $\mathbf{R}, \mathbf{t}$ from $\mathbf{E}$ [Pri12]

Relative orientation problem: recover  $\mathbf{R}, \mathbf{t}$  from  $\mathbf{E}$

- Essential matrix

$$\mathbf{E} = \mathbf{t}_\times \mathbf{R}$$

- To recover  $\mathbf{R}, \mathbf{t}$  use the matrix

$$\mathbf{W} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- With  $\mathbf{E} = \mathbf{UDV}^T$  we get:

$$\begin{aligned} \mathbf{t}_\times &= \mathbf{UDWU}^T \\ \mathbf{R} &= \mathbf{UW}^{-1}\mathbf{V}^T \end{aligned}$$

(details in [HZ04])

- Need 2 corresponding points to solve ambiguities and have  $\mathbf{R}$  and  $\mathbf{t}$  where points are in front of both cameras

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## Fundamental Matrix [Pri12]

- Lets consider two normal (not normalized) cameras:

$$\begin{aligned} \lambda_1 \tilde{\mathbf{x}}_1 &= \mathbf{K}_1 [\mathbf{I} | 0] \tilde{\mathbf{X}}_w \\ \lambda_2 \tilde{\mathbf{x}}_2 &= \mathbf{K}_2 [\mathbf{R} | \mathbf{t}] \tilde{\mathbf{X}}_w \end{aligned}$$

- Using a similar procedure we can get the relation:

$$\tilde{\mathbf{x}}_2^T \mathbf{K}_2^{-T} \mathbf{E} \mathbf{K}_1^{-1} \tilde{\mathbf{x}}_1 = 0$$

- or:

$$\tilde{\mathbf{x}}_2^T \mathbf{F} \tilde{\mathbf{x}}_1 = 0$$

- with:

$$\mathbf{F} = \mathbf{K}_2^{-T} \mathbf{E} \mathbf{K}_1^{-1} = \mathbf{K}_2^{-T} \mathbf{t}_\times \mathbf{R} \mathbf{K}_1^{-1}$$

- Relation between essential and fundamental matrix:

$$\mathbf{E} = \mathbf{K}_2^T \mathbf{F} \mathbf{K}_1$$

## Estimate the fundamental matrix [Pri12]

- When the fundamental matrix is correct with  $\tilde{\mathbf{x}}_{i2}^T \mathbf{F} \tilde{\mathbf{x}}_{i1} = 0$ , the epipolar line induced by a point in the first image should pass through the matching point in the second image and vice-versa
- Constraint parameterized by the nine entries of  $\mathbf{F}$
- Criterion: minimize the squared distance between every point and the epipolar line predicted by its match in the other image ( $I$  corresponding points):

$$\hat{\mathbf{F}} = \arg \min_{\mathbf{F}} \left[ \sum_{i=1}^I ((\text{dist}[\mathbf{x}_{i1}, l_{i1}])^2 + (\text{dist}[\mathbf{x}_{i2}, l_{i2}])^2) \right]$$

- $\text{dist}[\mathbf{x}, l] = \frac{ax+by+c}{\sqrt{a^2+b^2}}$  with  $l = [a, b, c]$  and  $\mathbf{x} = [x, y]^T$

**No closed form solution**

## Estimate the fundamental matrix: the eight-point algorithm [Pri12]

- Approach
  - ▶ solve for fundamental matrix using homogeneous coordinates
  - ▶ closed form solution (but don't minimize a geometric error but an algebraic error)
  - ▶ solution usually very close to the values that optimize the previous cost function
- In homogeneous coordinates:

$$\tilde{\mathbf{x}}_2^T \mathbf{F} \tilde{\mathbf{x}}_1 = 0 \Rightarrow \begin{bmatrix} x_{i2} & y_{i2} & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x_{i1} \\ y_{i1} \\ 1 \end{bmatrix} = 0$$

## Estimate the fundamental matrix: the eight-point algorithm [Pri12]

- Can be expressed as:  $\mathbf{A}\mathbf{f} = 0$  with  $\mathbf{f} = [f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}]^T$  and  $\mathbf{A}$  contains the combination of at least 8 pairs of points coordinates
- Find minimum of  $|\mathbf{A}\mathbf{f}|^2$  subject to  $|\mathbf{f}| = 1$
- Solution can be found by SVD of  $\mathbf{A}$ :  $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$ , setting  $\mathbf{f}$  to the last column of  $\mathbf{V}$
- Reform  $\mathbf{F}$  from  $\mathbf{f}$

## Estimate the fundamental matrix: the eight-point algorithm [Pri12]

- This procedure does not ensure that solution is rank 2.  
Solution: set last singular value to zero.
- Can be unreliable because of numerical problems to do with the data scaling → better to re-scale the data first
- Needs 8 points in general positions (cannot all be planar).
- Fails if there is not sufficient translation between the views

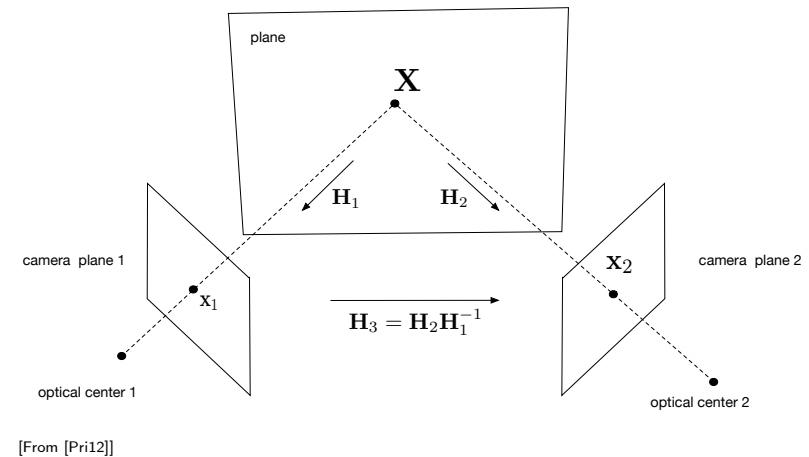
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## Homography: definition and properties



## Homography : definition and properties

- Homography mapping  $\tilde{x}_1$  and  $\tilde{x}_2$  linear in homogeneous coordinates

$$\lambda \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

- images seen by different cameras with the *pinhole in the same place* are related by homographies
- special case **pure rotation**: If the camera rotates but *does not translate*, the homography mapping image 1 to image 2 is written:

$$\mathbf{H} = \mathbf{K} \mathbf{R} \mathbf{K}^{-1}$$

with  $\mathbf{K}$  intrinsic matrix and  $\mathbf{R}$  rotation between the two camera positions

## Homography : definition and properties

Homography maps between [Pri12]:

- points on a plane in the world and their positions in an image,
- points in two different images of the same plane
- two images of a 3D object where the camera has rotated but not translated

In the planar case, we can chain the homographies between consecutive frames.

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## Homography estimation [HZ04]

Objective:

Given  $n \geq 4$  2D to 2D point correspondences  $\mathbf{x}'_i \leftrightarrow \mathbf{x}_i$ , determine the 2D homography matrix  $\mathbf{H}$  such as  $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$

Algorithm

- 1 **Normalization of  $\mathbf{x}_i$ :**  $\tilde{\mathbf{x}}_i = T\mathbf{x}_i$
- 2 **Normalization of  $\mathbf{x}'_i$ :**  $\tilde{\mathbf{x}}'_i = T'\mathbf{x}'_i$
- 3 **DLT:**
  - 1 For each correspondence  $\tilde{\mathbf{x}}'_i \leftrightarrow \tilde{\mathbf{x}}_i$  compute the matrix  $A_i$ .
  - 2 Form the  $2n \times 9$  matrix  $A$
  - 3 Write  $\tilde{\mathbf{h}}$  for the vector containing the entries of the matrix  $\tilde{\mathbf{H}}$ . A solution of  $A\tilde{\mathbf{h}} = 0$ , subject to  $\|\tilde{\mathbf{h}}\| = 1$ , is obtained from the unit singular vector of  $A$  corresponding to the smallest singular value.
  - 4 The matrix  $\tilde{\mathbf{H}}$  is determined from  $\tilde{\mathbf{h}}$
- 4 **Denormalization:** Set  $\mathbf{H} = T'^{-1}\tilde{\mathbf{H}}T$

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## From homography to pose computation

- All points are in the same plane, then fix  $z_w = 0$
- $\Rightarrow$  each 3D point coordinate is given by  $X_w = \begin{bmatrix} x_w \\ y_w \\ 0 \\ 1 \end{bmatrix}$
- Their projections in the image plane are given by

$$X_c = [\mathbf{I}|0][\mathbf{R}|\mathbf{t}] \begin{bmatrix} x_w \\ y_w \\ 0 \\ 1 \end{bmatrix} = [\mathbf{I}|0][r_1 r_2 \mathbf{t}] \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \mathbf{H} \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix}$$

with  $r_i$  the  $i^{th}$  column of  $\mathbf{R}$

- $\mathbf{H}$  can be computed using DLT

# From homography to pose computation

- Knowing  $\mathbf{H}$ ,  $\mathbf{R}$ ,  $\mathbf{t}$  can be computed noting:

$$[r_1 r_2 \mathbf{t}] = [\mathbf{I} | 0]^{-1} \mathbf{H}$$

- $\mathbf{R}$  is orthogonal then :  $r_3 = r_1 \times r_2$

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  - Corner detector
  - SIFT
  - Feature detectors
  - Outliers removal
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## Feature point matching principle

- Finding corresponding points (2D point in the image and a 3D reference or between two 2D images)
- Common framework:
  - keypoints extraction: subset of pixels ("cornerness")
  - description: conversion into a descriptor
  - matching between descriptors
- Common process:
  - off-line :
    - keypoint descriptors computed off-line on a reference model
    - key-point storage
  - on-line:
    - extract keypoint from each image
    - match in the descriptor space with those in the database
    - from correspondences, compute camera pose

## Feature points detectors properties

- Repeatability
- Invariance
- Robustness
- Distinctiveness/informativeness
- Locality
- Quantity
- Accuracy
- Efficiency

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## Corner detector

- Moravec [Mor80], Harris corner-detector [HS88], Shi-Tomasi [ST02]
- Harris corner-detector: Studies the average variation in intensity for a small movement

$$E(u, v) = \sum_{x,y} w(x, y) [I(x + u, y + v) - I(x, y)]^2$$

- ▶  $E$ : the difference between the original and the moved window.
- ▶  $u, v$ : window's displacement in the  $x$  (resp.  $y$ ) direction
- ▶  $w(x, y)$ : window function
- ▶  $I$ : image intensity at a position  $(x, y)$
- ▶  $I(x, y)$ : original intensity
- ▶  $I(x + u, y + v)$ : shifted intensity

## Corner detector

- Expansion using Talor series:

$$\begin{aligned} E(u, v) &\approx \sum_{x,y} w(x, y) [I(x, y) + uI_x + vI_y - I(x, y)]^2 \\ &\approx \sum_{x,y} w(x, y) [u^2 I_x^2 + 2uvI_x I_y + v^2 I_y^2] \end{aligned}$$

with  $I_x$  gradient along  $x$ ,  $I_y$  gradient along  $y$

- in matrix form

$$\begin{aligned} E(u, v) &\approx \begin{bmatrix} u & v \end{bmatrix} \left( \sum w(x, y) \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix} \\ &\approx \begin{bmatrix} u & v \end{bmatrix} M \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

- $M$ : symmetric, define, positive  $\Rightarrow$  eigenvalues decomposition



## Corner detector

- A score,  $R$ , is calculated for each window:

$$R = \det(M) - k(\text{trace}(M))^2$$

with  $\det(M) = \lambda_1 \lambda_2$  and  $\text{trace}(M) = \lambda_1 + \lambda_2$

- $R$  values are  $> 0$  around a corner,  $< 0$  around an edge and small in a constant region

Note: Shi-Tomasi detector has improved Harris detector

## Outline

- 1 Camera model and calibration
  - Feature point matching principle and properties
  - Corner detector
  - SIFT
  - Feature detectors
  - Outliers removal
- 2 Some pose estimation algorithms with a known 3D model
- 3 Transformation between images
  - Epipolar geometry
  - Homography
  - How to get correspondences?
- 4 Motion estimation
- 5 Conclusion
- 6 Links and bibliography

## SIFT [Low04]

- SIFT: Distinctive image features from scale invariant keypoint (1999) [Low04]
- has been considered a breakthrough for 2D points matching
- Robust to
  - ▶ Scale
  - ▶ Rotation
  - ▶ Illumination
  - ▶ Viewpoint

## SIFT [Low04] algorithm principle

- 1 Constructing a scale space
- 2 Laplacian of Gaussian (LoG) approximation using Difference of Gaussian (DoG)
- 3 Finding keypoints
- 4 Get rid of bad key points (edges and low contrast regions)
- 5 Assigning an orientation to the keypoints (cancels out the effect of orientation)
- 6 Generate SIFT features

A detailed explanation can be found in [ROD14].

## Outline

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## Feature detectors

- FAST [ST02]
- SIFT [Low04]
- SURF [BETV08]
- ORB [RRKB11]
- KAZE [ABD12]
- CARD [ABD12]
- BRIEF [CLÖ+12]
- BRISK [LCS11]
- FREAK [AOV12]
- LDB [XK12]
- ...

Demo : Learning OpenCV3 example code

[https://github.com/oreillymedia/Learning-OpenCV-3-examples/blob/master/example\\_16-02.cpp](https://github.com/oreillymedia/Learning-OpenCV-3-examples/blob/master/example_16-02.cpp)

## Outline

- 1 Camera model and calibration
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  - Corner detector
  - SIFT
  - Feature detectors
  - Outliers removal
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## Remove outliers

- Causes of outliers: image noise, occlusions, blur, changes in view point or illumination (non accounted by the feature detector)
- Use of RANSAC algorithm [FB81]
  - 1 Initial: let  $A$  be a set of  $N$  feature correspondences
  - 2 repeat
    - 1 Randomly select a sample of  $s$  points from  $A$
    - 2 Fit a model to these points
    - 3 Compute the distance of all other points to this model
    - 4 Construct the inlier set (i.e. count the number of points whose distance from the model  $< d$ )
    - 5 Store these inliers
    - 6 until maximum number of iterations reached
  - 3 The set with the maximum number of inliers is chosen as a solution to the problem
  - 4 Estimate the model using all the inliers

## Remove outliers

- The number of iterations  $N$  which ensures a probability  $p$  that at least one sample with only inliers is drawn is given by

$$N = \frac{\log(1 - p)}{\log(1 - (1 - \nu)^s)}$$

with

- ▶  $\nu$  the probability that a correspondence is an outlier
- ▶  $s$  the number of point from which the model can be instantiated
- Example:  $p=99\%$ ,  $s=5$ ,  $\nu=50\% \Rightarrow N=145$

## Outline

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- Homography
  - How to get correspondences ?
  - 3D reconstruction

## 3D reconstruction principle pipeline [Pri12]

- 1 Compute image features
- 2 Compute feature descriptors
- 3 Find initial matches
- 4 Compute fundamental matrix
- 5 Refine matches
- 6 Estimate essential matrix
- 7 Decompose essential matrix (four possibles solutions)
- 8 Estimate 3D points.

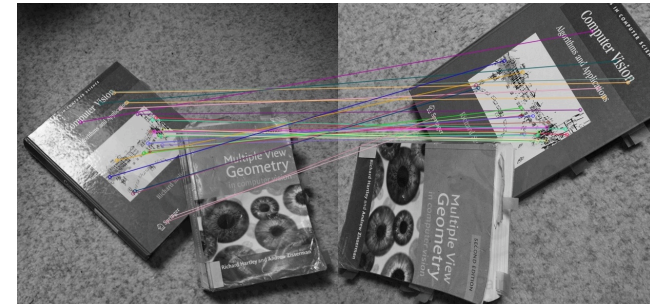
## Outline

- 1 Camera model and calibration
- 2 Some pose estimation algorithms with a known 3D model
- 3 Transformation between images
- 4 Motion estimation
  - Principle
  - Motion from feature correspondences
  - SLAM
- 5 Conclusion
- 6 Links and bibliography

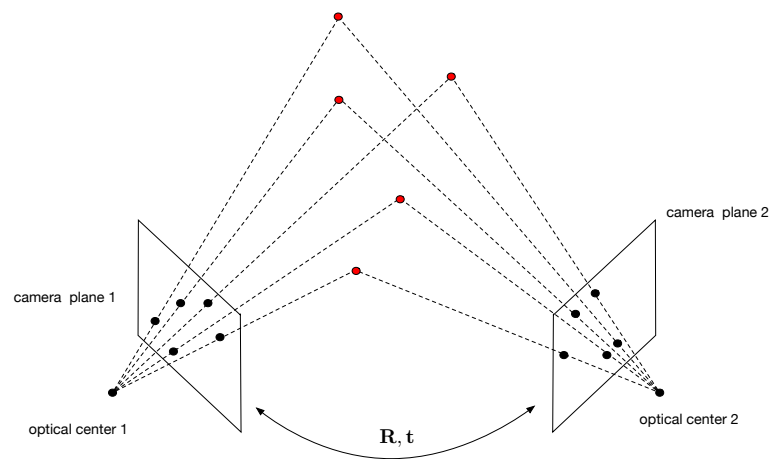
## Outline

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## Principle

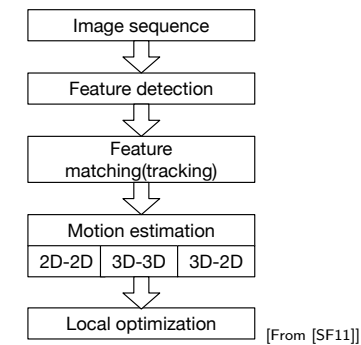


## Principle



## Principle

Visual Odometry (VO) compute the camera path incrementally

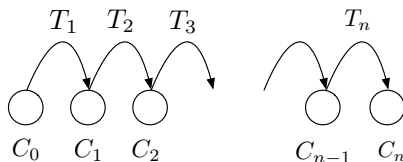


## Problem formulation [SF11]

- Two camera position at adjacent timestamps  $k - 1$  and  $k$  are related by

$$T_k = \begin{bmatrix} R_{k,k-1} & t_{k,k-1} \\ 0 & 1 \end{bmatrix}$$

- $T_{0:n} = T_1, \dots, T_n$  contents to all subsequent motions
- The set of camera poses  $C_{0:n} = C_0, \dots, C_n$  contains the transformation of the camera w.r.t.  $\mathcal{F}_{w0}$  at  $k = 0$
- $C_n = C_{n-1} T_n = C_0 T_1 \dots T_n$



## Problem formulation [SF11]

VO tasks:

- compute the relative transformations  $T_k$  from the images  $I_k$  and  $I_{k-1}$
- concatenate the transformations to recover the full trajectory  $C_{0:n}$  of the camera
- optionally perform an iterative refinement over last  $m$  poses to estimate local trajectory with more accuracy (bundle adjustment)

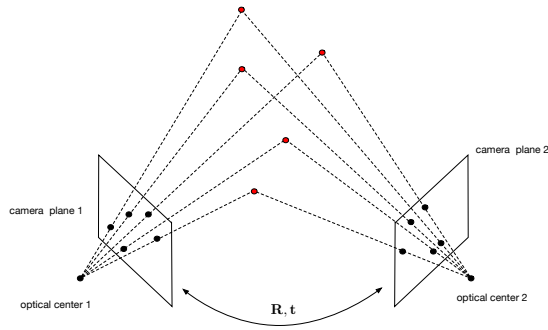
## Outline

- 1 Camera model and calibration
  - 2 Some pose estimation algorithms with a known 3D model
  - 3 Transformation between images
  - 4 Motion estimation
  - 5 Conclusion
  - 6 Links and bibliography
- Principle
  - Motion from feature correspondences
    - 2D-2D correspondences
    - 3D-3D correspondences
    - 3D-2D correspondences
  - SLAM

## Outline

- 1 Camera model and calibration
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    - 3D-3D correspondences
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  - SLAM

## 2D-2D [SF11]



- Both images features are specified in 2D
- The minimal-case solution involve 5-point correspondences
- The solution is found by determining the transformation that minimizes the reprojection error of the triangulated points in each image

## 2D-2D relative scale [SF11]

- Absolute scale of the translation cannot be computed from two images
- Triangulate 3D points position from 2D points pairs
- From 3D points, the relative distances between any combination of two 3-D points can be computed.
- Scale can then be determined from the distance ratio  $r$  between a point pair in  $X_{k-1}$  and a pair in  $X_k$

$$r = \frac{\|X_{k-1,i} - X_{k-1,j}\|}{\|X_{k,i} - X_{k,j}\|}$$

- Mean scale ratio is used to scale  $\mathbf{t}$

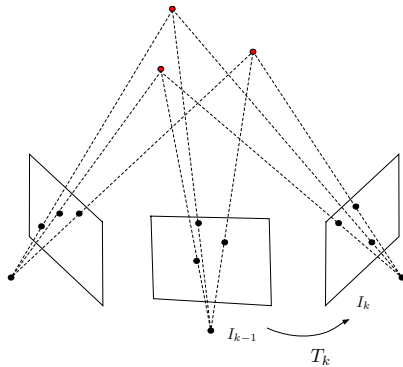
## Algorithm: VO for 2D-2D correspondence [SF11]

- 1 Capture new frame  $I_k$
- 2 Extract and match features between  $I_{k-1}$  and  $I_k$
- 3 Compute essential matrix for image pair  $I_{k-1}$  and  $I_k$
- 4 Decompose essential matrix into  $\mathbf{R}_k$  and  $\mathbf{t}_k$  and form  $T_k$
- 5 Compute relative scale and rescale  $\mathbf{t}_k$  accordingly
- 6 Concatenate transformation by computing  $C_k = C_{k-1} T_k$
- 7 Repeat from 1

## Outline

- 1 Camera model and calibration
- 2 Some pose estimation algorithms with a known 3D model
- 3 Transformation between images
- 4 Motion estimation
- 5 Principle
- 6 Motion from feature correspondences
  - 2D-2D correspondences
  - 3D-3D correspondences
  - 3D-2D correspondences
- 7 SLAM
- 8 Conclusion
- 9 Links and bibliography

## 3D-3D [SF11]



- Both image features are specified in 3D
- The minimal-case solution involve 3 non-linear correspondences
- The solution is found by determining the aligning transformation that minimizes the 3D-3D distance

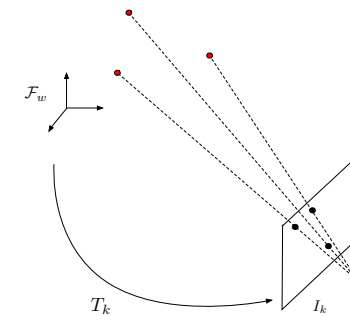
## Algorithm: VO for 3D-3D correspondence [SF11]

- 1 Capture two stereo image pairs  $I_{l,k-1}, I_{r,k-1}$  and  $I_{l,k}, I_{r,k}$
- 2 Extract and match features between  $I_{l,k-1}$  and  $I_{l,k}$
- 3 Triangulate matched features for each stereo pair
- 4 Compute  $T_k$  from 3-D features  $X_{k-1}$  and  $X_k$
- 5 Concatenate transformation by computing  $C_k = C_{k-1} T_k$
- 6 Repeat from 1

## Outline

- Principle
- Motion from feature correspondences
  - 2D-2D correspondences
  - 3D-3D correspondences
  - 3D-2D correspondences
- SLAM
- 5 Conclusion
- 6 Links and bibliography
- 1 Camera model and calibration
- 2 Some pose estimation algorithms with a known 3D model
- 3 Transformation between images
- 4 Motion estimation

## 2D-3D [SF11]

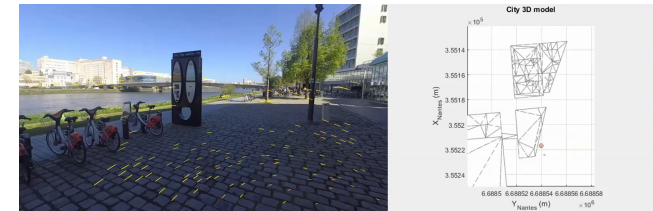


- Features at instant  $k - 1$  are in 3D and in 2D at  $k$
- PnP problem
- The minimal-case solution involve 3 non-linear correspondences
- The solution is found by determining the transformation that minimizes the reprojection error

## Algorithm: VO for 2D-3D correspondence [SF11]

- 1 Do only once:
  - 1 Capture two frames  $I_{k-2}, I_{k-1}$
  - 2 Extract and match features between them
  - 3 Triangulate features from  $I_{k-2}, I_{k-1}$
- 2 Do at each iteration:
  - 1 Capture new frame  $I_k$
  - 2 Extract features and match with previous frame  $I_{k-1}$
  - 3 Compute camera pose (PnP) from 3-D-to-2-D matches
  - 4 Triangulate all new feature matches between  $I_k$  and  $I_{k-1}$
  - 5 Iterate

## VO trajectory estimation example



work from [ASR18]

## Outline

- 1 Camera model and calibration
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  - Principle
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  - SLAM
    - VO vs. SLAM
    - vSLAM and viSLAM
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## VO vs. SLAM

- SLAM: *Simultaneous Localization And Mapping*
  - ▶ global consistent estimate of the localization (and mapping)
  - ▶ use loop-closure to reduce the drift in the map and in the localization of the camera (global bundle adjustment)
- VO
  - ▶ incremental localization
  - ▶ local consistent estimate of the trajectory
  - ▶ potential windowed bundles adjustments
- VO can be a part of SLAM (before closing the loop)

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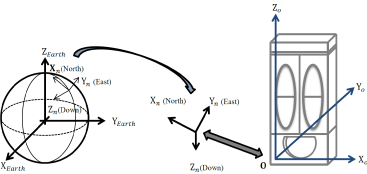
## vSLAM and viSLAM algorithms

- Visual SLAM
  - ▶ PTAM [KM07]
  - ▶ ORB-SLAM [MAMT15]
  - ▶ DTAM [NLD11]
  - ▶ LSD-SLAM [ESC14]
  - ▶ DSO [EKC16]
  - ▶ ...
- Visual inertial SLAM
  - ▶ MSCKF [MR07]
  - ▶ ROVIO [BOHS15]
  - ▶ S-MSCKF [SMP<sup>+</sup>17]
  - ▶ VINS-Mono [QLS17]
  - ▶ ...

## Outline

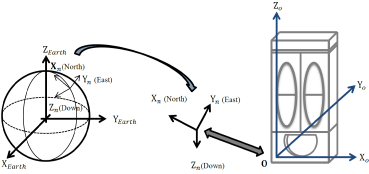
- 3 Transformation between images
- 1 Camera model and calibration
- 2 Some pose estimation algorithms with a known 3D model
- 4 Motion estimation
- 5 Conclusion
- 6 Links and bibliography

Conclusion



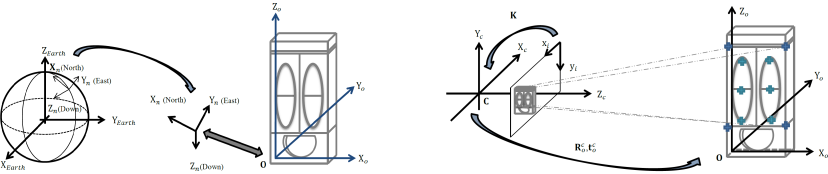
Images from [ASR17]

Conclusion



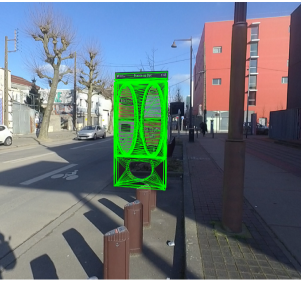
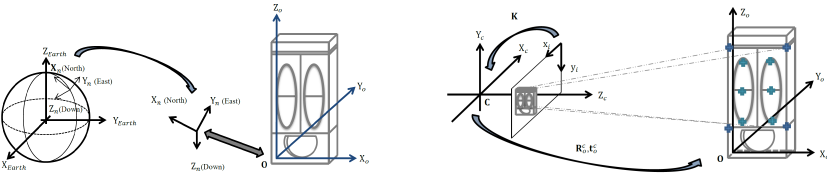
Images from [ASR17]

Conclusion



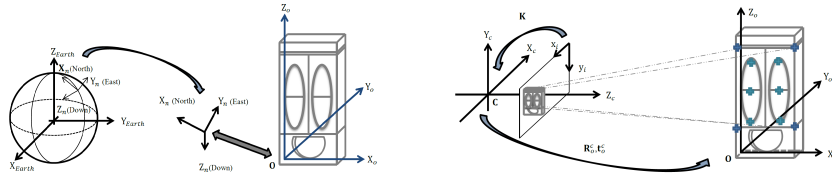
Images from [ASR17]

Conclusion



Images from [ASR17]

## Conclusion



Images from [ASR17]

## Conclusion

- But... pose calculation is not very accurate in the presence of fast rotation movements
- ... fusion with IMMU can help!

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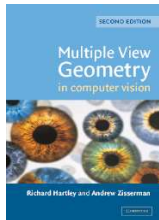
## Links

- Tools for Computer Vision
  - ▶ OpenCV <https://opencv.org> (Open source)
  - ▶ OpenGV <https://laurentkneip.github.io/opengv/index.html> (Open source)
  - ▶ ViSP <https://visp.inria.fr> (Open source)
  - ▶ MATLAB Toolboxes <https://fr.mathworks.com/solutions/image-video-processing.html>
- Websites
  - ▶ Annotated Computer Vision Bibliography <http://www.visionbib.com/bibliography/contents.html>
  - ▶ Computer Vision conferences list <http://conferences.visionbib.com/Iris-Conferences.html>

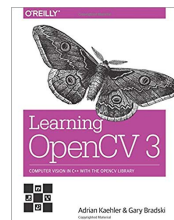
## (very) Useful books and tutorials

- **Multiple View Geometry in Computer Vision** [HZ04]  
From Richard Hartley, Andrew Zisserman

► [Book site](#)



- **Learning OpenCV 3: Computer Vision in C++ with the OpenCV Library** [KB17]  
From Adrian Kaehler and Gary Bradski



## (very) Useful books and tutorials

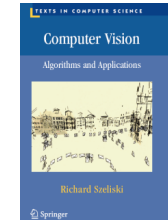
- **Computer Vision: Models Learning and Inference** [Pri12]  
From Simon J.D. Prince,

► <http://www.computervisionmodels.com>



- **Computer Vision: Algorithms and Applications** [Sze10]  
From Richard Szeliski

► <http://szeliski.org/Book/>



## (very) Useful books and tutorials

- E. Marchand, H. Uchiyama, F. Spindler "Pose estimation for augmented reality : a hands-on survey", IEEE Trans. Vis. Comput. Graph., 2016. [MUS16]
- D. Scaramuzza and F. Fraundorfer, "Visual Odometry: Part I - The First 30 Years and Fundamentals," IEEE Robot. Autom. Mag., vol. 18, no. 4, pp. 80-92, Dec. 2011. [SF11]
- F. Fraundorfer and D. Scaramuzza, "Visual Odometry : Part II: Matching, Robustness, Optimization, and Applications," IEEE Robot. Autom. Mag., vol. 19, no. 2, pp. 78-90, Jun. 2012. [FS12]

► <http://people.rennes.inria.fr/Eric.Marchand/pose-estimation/index.html>

► [http://rpg.ifi.uzh.ch/visual\\_odometry\\_tutorial.html](http://rpg.ifi.uzh.ch/visual_odometry_tutorial.html)

**So now, if you know what you look at, you can tell me where you are.**

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
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